Bode Plots & Network Frequency Response

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Transfer Function $H(\omega)$

The transfer function $H(\omega)$ is the frequency-dependent ratio of a phasor output $Y(\omega)$ [typically a voltage or a current] to a phasor input $X(\omega)$ [source voltage or current].

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$H(\omega) = \text{voltage gain} = \frac{V_O(\omega)}{V_I(\omega)}$  \quad  $H(\omega) = \text{current gain} = \frac{I_O(\omega)}{I_I(\omega)}$

$H(\omega) = \text{transfer impedance} = \text{transimenedance} = \frac{V_O(\omega)}{I_I(\omega)}$

$H(\omega) = \text{transfer admittance} = \frac{I_O(\omega)}{V_I(\omega)}$
Example: Transfer Function $H(\omega)$

Current gain transfer function $= \frac{I_0(\omega)}{I_i(\omega)}$

$I_0(\omega) = \frac{4 + j2\omega}{4 + j2\omega + \frac{1}{j0.5\omega}} I_i(\omega)$ \text{ or }$

\frac{I_0(\omega)}{I_i(\omega)} = \frac{j0.5\omega(4 + j2\omega)}{4 + j2\omega + (j\omega)^2} = \frac{s(s + 2)}{s^2 + 2s + 1}$ \text{ where } \ s = j\omega

This is a ratio of polynomials which can be factored.

The zeros are at $s(s+2) = 0 \implies z_1 = 0, z_2 = -2$

The poles are at $s^2 + 2s + 1 = (s + 1)(s + 1) = 0$

There is a double pole at $p_{1,2} = -1$
Example: R-L Circuit

\[ Ri(t) + L \frac{di(t)}{dt} = v(t) ; \quad \text{Let} \quad i(t) = I e^{st} \quad \text{and} \quad v(t) = V e^{st} \]

Therefore,

\[ (R e^{st} + s L e^{st}) I = V e^{st} \]

\[ H(s) = \frac{i(t)}{v(t)} = \frac{I}{V} = \frac{1}{R + sL} \]

So,

\[ I e^{st} = H(s) V e^{st} \]
Example: R-L Circuit

\[ H(s) = \frac{i(t)}{v(t)} = \frac{I}{V} = \frac{1}{R + sL} = \frac{1}{R} \left( 1 + \frac{s}{R/L} \right) \]

where \( 2\pi f_p = \frac{R}{L} = \omega_p \)

These are log-log plots – frequency is logarithmic
Example: R-C Circuit

\[ i(t) = \frac{v(t)}{R} + C \frac{dv(t)}{dt} ; \quad \text{Let } i(t) = Ie^{st} \text{ and } v(t) = Ve^{st} \]

Therefore,

\[
\left( \frac{e^{st}}{R} + sC \cdot e^{st} \right) V = Ie^{st}
\]

\[
H(s) = \frac{\frac{v(t)}{i(t)}}{\frac{1}{I}} = \frac{1}{\left( \frac{1}{R} + sC \right)} = \frac{R}{1 + sRC}
\]

So,

\[ Ve^{st} = H(s)Ie^{st} \]
Examples of Bode plots for 1\textsuperscript{st} order circuits

(a) High-pass circuit

(b) Low-pass circuit
General Form of $H(s)$

Transfer Function $H(s)$

$$H(s) = \frac{b_n}{a_m} \cdot \frac{\left(1 + \frac{s}{\omega_{z1}}\right) \left(1 + \frac{s}{\omega_{z2}}\right) \cdots \left(1 + \frac{s}{\omega_{zn}}\right)}{\left(1 + \frac{s}{\omega_{p1}}\right) \left(1 + \frac{s}{\omega_{p2}}\right) \cdots \left(1 + \frac{s}{\omega_{pm}}\right)}$$

We must factor polynomials in variable $s$ where $s = j\omega$
R-L and R-C Circuits

Low-pass

$$T = CR$$
$$\omega_c = \frac{1}{T} = \frac{1}{CR}$$
$$f_c = \frac{\omega_c}{2\pi} = \frac{1}{2\pi CR}$$

High-pass

$$T = \frac{L}{R}$$
$$\omega_c = \frac{1}{T} = \frac{R}{L}$$
$$f_c = \frac{\omega_c}{2\pi} = \frac{R}{2\pi L}$$
Some logarithm values convenient in circuit analysis

| $|H(s)|$ | $|H(s)|_{dB}$ |
|------|--------------|
| 1    | $20 \log(1) = 0 \text{ dB}$ |
| $\sqrt{2}$ | $20 \log(\sqrt{2}) = 3 \text{ dB}$ |
| 2    | $20 \log(2) = 6 \text{ dB}$ |
| 4    | $20 \log(4) = 12 \text{ dB}$ |
| 5    | $20 \log(5) = 14 \text{ dB}$ |
| 10   | $20 \log(10) = 20 \text{ dB}$ |
| $1/\sqrt{2}$ | $20 \log(1/\sqrt{2}) = -3 \text{ dB}$ |
| $1/2$ | $20 \log(1/2) = -6 \text{ dB}$ |
| $1/4$ | $20 \log(1/4) = -12 \text{ dB}$ |
| $1/5$ | $20 \log(1/5) = -14 \text{ dB}$ |
| $1/10$ | $20 \log(1/10) = -20 \text{ dB}$ |
Low-pass Filter R-C Network’s Bode Plot

- Cutoff frequency
- Slope: -20 dB/decade
- -3.01 dB
Example

Suppose we have \( H(s) = \frac{A(S + z_1)}{(S + p_1)} \), then

\[
|H(s)|_{dB} = 20 \log |A| + 20 \log |S + z_1| - 20 \log |S + p_1|
\]

Example: \( H(s) = \frac{10(s + 100)}{(s + 1)} \)

A = 10

Pole at \( s = -1 \)

Zero at \( s = -100 \)

Now add all three together
All together now

\[ H(s) = \frac{10(s+100)}{(s+1)} \]

\[ \lim_{s \to 0} |H(s)| = 60 dB \]

\[ \lim_{s \to \infty} |H(s)| = 20 dB \]
Phase in Bode Plots

The transfer function $H(s)$ is a phasor.
For a rational function $H(s)$ we add the phases from the numerator and subtract the phases from the denominator:

$$H(j\omega) = \frac{N(j\omega)}{D(j\omega)} \quad \Rightarrow \quad \angle H(j\omega) = \angle N(j\omega) - \angle D(j\omega)$$

Example (one zero & one pole):

$$H(j\omega) = \frac{(j\omega + z)}{(j\omega + p)} \quad \Rightarrow \quad \angle H(j\omega) = \tan^{-1}\left(\frac{\omega}{z}\right) - \tan^{-1}\left(\frac{\omega}{p}\right)$$
The capacitive load, $C_L$, is the cause of gain roll-off since at high frequency, it “steals” away some signal current and shunt it to ground.

\[
V_{out} = -g_m V_{in} \left( R_D \parallel \frac{1}{C_L s} \right) = \frac{-g_m R_D}{(1 + sR_D C_L)} \times V_{in}
\]
The circuit only has one pole (no zero) at \( \frac{1}{R_D C_L} \), so the slope drops from 0 to -20dB/dec as we pass frequency \( \omega_{p1} \).

\[ |\omega_{p1}| = \frac{1}{R_D C_L} \]
Example of a Circuit with Two Poles

\[ |\omega_{p1}| = \frac{1}{R_S C_{in}} \]

\[ |\omega_{p2}| = \frac{1}{R_D C_L} \]

\[ \left| \frac{V_{out}}{V_{in}} \right| = \frac{g_m R_D}{\sqrt{(1 + \omega^2/\omega_{p1}^2)(1 + \omega^2/\omega_{p2}^2)}} \]
Example of a Circuit with Two Poles

\[ \frac{v_{gs}}{sC_{in}} = v_{in} \left( R_s + \frac{1}{sC_{in}} \right) \quad \Rightarrow \quad \frac{v_{gs}}{v_{in}} = \frac{1}{1 + sR_s C_{in}} \]

\[ v_{out} = \left( -g_m v_{gs} \right) \times \frac{R_D}{1 + sR_D C_L} \quad \text{and substituting for } v_{gs} \]

\[ v_{out} = \left( -g_m \right) \times \frac{1}{1 + sR_s C_{in}} \times \frac{R_D}{1 + sR_D C_L} v_{in} \]

\[ \frac{v_{out}}{v_{in}} = \frac{-g_m R_D}{(1 + sR_s C_{in}) \times (1 + sR_D C_L)} \quad \text{(2 poles)} \]
Origin of Capacitances in MOSFET – I

- For a MOS, there exist oxide capacitance from gate to channel, junction capacitances from source/drain to substrate, and overlap capacitance from gate to source/drain.
The gate oxide capacitance is often partitioned between source and drain. In saturation, $C_2 \sim C_{\text{gate}}$, and $C_1 \sim 0$. They are in parallel with the overlap capacitance to form $C_{GS}$ and $C_{GD}$.
Origin of Capacitances in BJT – I

- At high frequency, capacitive effects come into play. $C_b$ represents the base charge, whereas $C_\mu$ and $C_{je}$ are the junction capacitances.

$$C_\pi = C_b + C_{je}$$
Origin of Capacitances in BJT – II

- Since an integrated bipolar circuit is fabricated on top of a substrate, another junction capacitance exists between the collector and substrate, namely $C_{CS}$. 

[Diagrams of a transistor schematic and capacitance symbols]
Example of Capacitors in BJT Circuit

Summary

• The frequency response refers to the magnitude of the transfer function.
• Bode’s approximation simplifies the plotting of the frequency response if pole and zero locations are known.
• In general, we can associate a pole with each node in the signal path.
• Bipolar and MOS devices possess various capacitances due to their physical structure that limit the speed of circuits.