Problem 1 (10 points)

How many possible combinations of answers to a 10-question true/false test are there? Assume that being correct is not a requirement in answering any of the questions.

Solution:

Each of the ten questions in the true/false test has only two possible answers. There are ten questions so the number of combinations of answers to ten questions is given by

\[ 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^{10} = 1,024 \text{ combinations} \]

Problem 2 (10 points)

A student has a combination lock requiring three numbers, ranging from 1 to 9, subject to the condition that a number in the combination can’t be repeated. How many different lock combinations are possible?

Solution:

The combination consists of a sequence of three numbers. For the first number there are 9 possible choices. For the second number (no repetition allowed) there are 8 possible choices and for the third number there are 7 choices. We multiply 9 times 8 times 7 to get the total number of combinations possible.

Number of combinations = \( 9 \times 8 \times 7 = 504 \text{ combinations} \)
Problem 3 (10 points)

How many different ways can you rearrange the letters in the word, BOSTON? (Hint: There are six letters but two of them are duplicates.)

Solution:

If all six letters were unique (that is, no repetition) then there would be $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$ Distinct ways to rearrange the letters. However, two of the letters repeat, so we must divide by $2! = 2$ to find the correct number of ways.

$$\text{Number of ways to rearrange letters} = \frac{6!}{2!} = \frac{720}{2} = 360$$

For the rest of this problem we must work with ordered without repetition to be consistent.

Problem 4 (25 points)

There are $6^4 = 1,296$ possible outcomes when rolling four dice (combinations, not permutations). Count how many of these outcomes occur for each of the conditions below:

(a) No pairs?

Solution:

Each number must be unique for each of the values of the four dice. The first die considered can have any one of the six possible face values of a die. The second die has five possible values to select from (one has already been taken), and the third die has four remaining possible values. Finally, the fourth die can assume one of the three remaining values.

$$6 \times 5 \times 4 \times 3 = 360 \text{ combinations}$$

(b) Exactly one pair?

Solution:
There are $\binom{4}{2}$ choices for which two dice have the same value: 6 choices for the two dice forming the pair ($6 \times 1$). Then third die can take one of the five values ($6 - 1 = 5$ values left) and the fourth die has only four values it can take on.

$$6 \times 1 \times 5 \times 4 = 120 \text{ in total. But wait because . . .}$$

However, we must consider the ordered aspect of having one pair. How many ways can one pair be ordered? There are six ways to order one pair as illustrated below: (Note: white squares can be any other number.)

Thus, there are six ways to order one pair, so we multiply 120 by 6 to get

$$6 \times (6 \times 1 \times 5 \times 4) = 720 \text{ in total}$$

(c) Exactly two pairs?

**Solution:**

The key idea is that there are 6 choices for the common value of the first pair and 5 choices for the common value of the second pair. However, there are 6 ways to order two pair (see above figure with blue being one of the two pair and white being the other pair). This gives us

$$6 \times (6 \times 1 \times 5 \times 1) = 180 \text{ in total}$$
Now, if we were to determine the number of combinations for a triple and finally, four identical numbers (with 6 combinations), they will all total 1,296 in total.

**Problem 5** (25 points)

How many ways can three women and three men be seated on six chairs?

(a) In a row of chairs with no seating restrictions?

**Solution:**

There are $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720$

(b) If women and men must alternate with the chairs in a row?

**Solution:**

For the first chair there are 6 choices (it could be either a woman or a man), but once chosen, then three choices for the second chair from the opposite sex; for the third chair there are 2 choices available; for the fourth chair there are two choices; finally, only one choice remains for the las two chairs.

$$6 \times 3 \times 2 \times 2 \times 1 \times 1 = 72 \text{ permutations}$$

(c) If the chairs are now arranged in a circle under the restriction that no two men can sit next to each other?

**Solution:**

For the first chair there are 6 choices (and it could be either a man or woman); for the second chair it must be a person of the opposite sex so, there are only 3 choices; for the third chair, and again alternating there are 2 choices; for the fourth chair there are 2 choices; finally, the last two
Problem 6 (10 points)

Ten people attend a potluck dinner. Each of the ten people are assigned what to bring as a dish.
- 5 people are to bring a main dish
- 3 people are to bring drinks
- 2 people are to bring a dessert

How many ways can ten people be assigned what dish to bring?

Solution:

Ten people are available to bring 5 main dishes; then five people are available to bring three drinks; finally, two people are assigned two Desserts. We use the binomial coefficient to determine the number of combinations (unordered sampling of distinguishable items without replacement) and multiply them together as shown below.

\[
\binom{10}{5} \times \binom{5}{3} \times \binom{2}{2} = \frac{10!}{5!5!} \times \frac{5!}{3!2!} \times \frac{2!}{2!0!} = \frac{10!}{5!3!2!} = 2,520
\]

Problem 7 (10 points)

Numerically evaluate the following binomial coefficients:

(a) \( \binom{12}{8} \)
Solution:
\[
\binom{12}{8} = \frac{12!}{8!4!} = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) \times (4 \times 3 \times 2 \times 1)} = \frac{12 \times 11 \times 10 \times 9}{24} = 495
\]

(b) \( \binom{18}{2} \)

Solution:
\[
\binom{18}{2} = \frac{18!}{2!16!} = \frac{18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(2 \times 1) \times (16 \times 15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1)}
\]
\[
= \frac{18 \times 17}{2} = 153
\]