Problem 1 (15 points)

Given the probability mass function \( P_V(v) = c v^2 \) for \( v = 1, 2, 3, 4 \) and \( P_V(v) = 0 \) for all other integer values of \( v \).

(a) Find the constant \( c \).

Solution:

Use the relationship: \( P_V(1) + P_V(2) + P_V(3) + P_V(4) = 1 \). Then we have
\[
c(1) + c(2^2) + c(3^2) + c(4^2) = c(1 + 4 + 9 + 16) = 30c = 1,
\]
Therefore, \( c = 1/30 \)

(b) Find the probability that random variable \( V \) is an even number.

Solution:

In this case \( P_V(2) + P_V(4) = (1/30)(4 + 16) = 2/3 \)

(c) Find \( P_V(V > 2) \).

Solution:

In this case \( P_V(3) + P_V(4) = (1/30)(9 + 16) = 5/6 \)

Problem 2 (25 points)

You have a single six-faced die with faces numbered 1 through 6. Each outcome possible with a roll of the die has equal probability and let the random
variable $X$ represent the face up value of the die after it comes to rest upon being thrown. Therefore, the probability mass function is given by $P_X(x_k) = 1/6$ for $k = 1, 2, 3, 4, 5, 6$.

(a) Plot the probability mass function on the graph below. Be sure to label the abscissa and ordinate of the graph (with values indicated). Note: The PMF is a discrete distribution known as the UNIFORM distribution.

![Graph showing the probability mass function for a die roll]

Solution:
This is a discrete uniform distribution.

(b) Calculate the expectation value $E[X] = \mu_X$ for this uniform distribution. Remember that $E[X] = \sum_{k=1} \frac{1}{6} x_k \cdot P_X(X = x_k)$ for all values of $k$. 

$E[X] = \mu_X = 3.5$
Solution:

We can write the following expression,

\[ E[X] = x_1 \cdot P_x(1) + x_2 \cdot P_x(2) + x_3 \cdot P_x(3) + x_4 \cdot P_x(4) + x_5 \cdot P_x(5) + x_6 \cdot P_x(6) \]

\[ E[X] = \frac{1}{6} \times (1) + \frac{1}{6} \times (2) + \frac{1}{6} \times (3) + \frac{1}{6} \times (4) + \frac{1}{6} \times (5) + \frac{1}{6} \times (6) \]

\[ E[X] = \frac{1}{6} \times (1 + 2 + 3 + 4 + 5 + 6) = \frac{21}{6} = 3.5 = \mu_X \]

(c) Calculate the Variance \( E[(X - \mu_X)^2] = \sigma_X^2 \) for the uniform distribution.
Remember that \( E[(X - \mu_X)^2] = \sum_{k=1}^{6} (x_k - \mu_X)^2 \cdot P_x(X = x_k) \) for all values of \( k \).

Solution:

We can write the following expression,

\[ E[(X - \mu_X)^2] = (x_1 - \mu_X)^2 \cdot P_x(1) + (x_2 - \mu_X)^2 \cdot P_x(2) + (x_3 - \mu_X)^2 \cdot P_x(3) + (x_4 - \mu_X)^2 \cdot P_x(4) + (x_5 - \mu_X)^2 \cdot P_x(5) + (x_6 - \mu_X)^2 \cdot P_x(6) \]

\[ E[(X - \mu_X)^2] = (1 - 3.5)^2 \times \frac{1}{6} + (2 - 3.5)^2 \times \frac{1}{6} + (3 - 3.5)^2 \times \frac{1}{6} + (4 - 3.5)^2 \times \frac{1}{6} + (5 - 3.5)^2 \times \frac{1}{6} + (6 - 3.5)^2 \times \frac{1}{6} \]

\[ E[(X - \mu_X)^2] = (-2.5)^2 \times \frac{1}{6} + (-1.5)^2 \times \frac{1}{6} + (-0.5)^2 \times \frac{1}{6} + (0.5)^2 \times \frac{1}{6} + (1.5)^2 \times \frac{1}{6} + (2.5)^2 \times \frac{1}{6} \]

\[ E[(X - \mu_X)^2] = \frac{1}{6} \times (6.25 + 2.25 + 0.25 + 0.25 + 2.25 + 6.25) \]

\[ = \frac{17.5}{6} = 2.917 \]

\( \sigma_X = \sqrt{E[(X - \mu_X)^2]} = \sqrt{2.917} = 1.708 \)
Problem 3 (15 points)

On the average, one computer out of 800 computers crashes during a severe thunderstorm. Company XYZ has 4,000 operational computers on site when a severe thunderstorm strikes. Find the following:

Compute the expected value and variance of the number of crashed computers from the thunderstorm.

Solution:

Let $X$ be the number of crashed computers. This is number of “successes” (i.e., crashed computers) out of the total of 4,000 “trials” (computers), with the probability of success being $1/800$. Thus, we are dealing with a binomial distribution with parameters $n = 4,000$ and $p = 1/800 (= 0.00125)$.

Using the binomial distribution with $n = 4,000$ and $p = 0.00125$.

$$E[X] = np = 5$$  and  $$Var[X] = np(1 - p) = 5 \times \left(1 - \frac{1}{800}\right) = 4.994$$

This distribution is approximately Poisson with parameter $\lambda = np = 5$.

Problem 4 (20 points)

50 students live in a dormitory. The parking lot has the capacity for 30 cars. If each student has a car with probability $1/2$ (independently from other students); what is the probability that there won't be enough parking spaces for all the cars? DO NOT EVALUATE THE SERIES OF BINOMIAL COEFFICIENTS.

Solution:

Let $X$ be the number of cars owned by the 50 students. This is a binomial distribution with parameters of $n = 50$ and $p = \frac{1}{2}$. 
\[ P(X > 30) = \sum_{k=31}^{50} \binom{50}{k} \left( \frac{1}{2} \right)^k \left( \frac{1}{2} \right)^{50-k} = \sum_{k=31}^{50} \binom{50}{k} \left( \frac{1}{2} \right)^{50} \]

\[ P(X > 30) = \left( \frac{1}{2} \right)^{50} \sum_{k=31}^{50} \binom{50}{k} \]

**Problem 5 (15 points)**

The number of adults living in homes on a randomly selected city district is described by the probability distribution in the table.

<table>
<thead>
<tr>
<th>Number of adults ( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability ( P_X(x) )</td>
<td>0.20</td>
<td>0.43</td>
<td>0.20</td>
<td>0.09</td>
<td>0.05</td>
<td>0.06</td>
</tr>
</tbody>
</table>

Find the standard deviation of the probability distribution.

**Solution:**

The mean value is calculated by

\[ \text{Mean } \mu_x = 1 \times 0.20 + 2 \times 0.43 + 3 \times 0.20 + 4 \times 0.09 + 5 \times 0.05 + 6 \times 0.03 = 2.45 \quad \Leftarrow \]

The variance is calculated by

\[ \text{Var}[X] = (1 - 2.45)^2 \times 0.20 + (2 - 2.45)^2 \times 0.43 + (3 - 2.45)^2 \times 0.20 + (4 - 2.45)^2 \times 0.09 \]
\[ + (5 - 2.45)^2 \times 0.05 + (6 - 2.45)^2 \times 0.03 \]
\[ = 0.421 + 0.087 + 0.061 + 0.216 + 0.325 + 0.378 = 1.488 \quad \Leftarrow \]

\[ \sigma_x^2 = \text{Var}[X]; \text{ hence, } \sigma_x = \sqrt{1.488} = 1.220 \quad \Leftarrow \]

**Problem 6 (25 points)**

The histogram of the final exam in an ES 101A class from 2014 is shown below. The total number of students taking the exam was 74 students. Estimate the mean value (expectation value) of this data distribution.
Solution:

The mean value for the exam is found by summing up the products of the score multiplied by the number of students with that score. For a bar take the midpoint for the range of the score (example: for the bar going from 80 to 90, assume the mid-score is 85.

\[
\text{Mean} = \frac{[1 \times 35 + 1 \times 55 + 1 \times 65 + 3 \times 75 + 8 \times 85 + 9 \times 95 + 17 \times 105 + 10 \times 115 + \\
+ 14 \times 125 + 5 \times 135 + 5 \times 145]}{74} \\
= \frac{8,000}{74} = 108.1
\]