Problem 1 (48 points)

Given continuous random variable $X$ and $Y$ with joint PDF of

$$f_{XY}(x, y) = \begin{cases} c(xy^2 + x^2y) & \text{for } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant $c$.

$$1 = \int_0^1 \int_0^1 c(xy^2 + x^2y) dy dx = \int_0^1 c \left( \frac{x}{3} + \frac{x^2}{2} \right) dx = c \left( \frac{1}{6} + \frac{1}{6} \right) = \frac{c}{3}$$

Thus, $c = 3 \iff$

(b) Find the joint CDF $F_{XY}(x, y)$ for the above PDF.

$$F_{XY}(x, y) = P(X \leq x, Y \leq y) = 3 \int_0^x \int_0^y (uv^2 + u^2v) dv du$$

$$= 3 \int_0^x \left[ \frac{uv^3}{3} + \frac{u^2y}{2} \right] du = \frac{x^2y^3 + x^3y^2}{2}$$

(c) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

$$f_X(x) = \int_0^1 f_{XY}(x, y) dy = \int_0^1 3(xy^2 + x^2y) dy = 3 \left( \frac{x}{3} + \frac{x^2}{2} \right)$$

$$f_Y(y) = \int_0^1 f_{XY}(x, y) dx = \int_0^1 3(xy^2 + x^2y) dx = 3 \left( \frac{y^2}{2} + \frac{y}{3} \right)$$
(d) Find the marginal CDFs \( F_X(x) \) and \( F_Y(y) \).

\[
F_X(x) = F_{XY}(x, y = 1) = \frac{x^2 y^3 + x^3 y^2}{2} \bigg|_{y=1} = \frac{x^2 + x^3}{2}
\]

\[
F_Y(y) = F_{XY}(x = 1, y) = \frac{x^2 y^3 + x^3 y^2}{2} \bigg|_{x=1} = \frac{y^3 + y^2}{2}
\]

(e) Find the expectation values \( EX \) and \( EY \).

\[
EX = \int_0^1 x f_X(x) dx = \int_0^1 3x \left( \frac{x}{3} + \frac{x^2}{2} \right) dx = 3 \left( \frac{1}{9} + \frac{1}{8} \right) = \frac{17}{24} \approx 0.7083
\]

\[
EY = \int_0^1 y f_Y(y) dy = \int_0^1 3y \left( \frac{y^2}{2} + \frac{y^3}{3} \right) dy = 3 \left( \frac{1}{8} + \frac{1}{9} \right) = \frac{17}{24} \approx 0.7083
\]

(f) Find the variances \( \text{Var}(X) \) and \( \text{Var}(Y) \). Use \( \text{Var}(X) = E(X^2) - (E(X))^2 \) to do the calculations. Express numerically.

We start with calculating \( E(X^2) \) and \( E(Y^2) \).

\[
E(X^2) = \int_0^1 x^2 f_X(x) dx = \int_0^1 3x^2 \left( \frac{x}{3} + \frac{x^2}{2} \right) dx = 3 \left( \frac{1}{12} + \frac{1}{10} \right) = \frac{11}{20} \approx 0.5500
\]

\[
E(Y^2) = \int_0^1 y^2 f_Y(y) dy = \int_0^1 3y^2 \left( \frac{y^2}{2} + \frac{y^3}{3} \right) dy = 3 \left( \frac{1}{10} + \frac{1}{12} \right) = \frac{11}{20} \approx 0.5500
\]

From

\[
\text{Var}(X) = E(X^2) - (E(X))^2 = 0.5500 - (0.7083)^2 = 0.04826
\]

and

\[
\text{Var}(Y) = E(Y^2) - (E(Y))^2 = 0.5500 - (0.7083)^2 = 0.04826
\]

(g) Find the standard deviations of \( X \) and \( Y \).

Start with the relationship \( \sigma^2 = \text{Var}(X) \),

\[
\sigma_X = \sqrt{\text{Var}(X)} = \sqrt{0.04825} = 0.2197
\]

\[
\sigma_Y = \sqrt{\text{Var}(Y)} = \sqrt{0.04825} = 0.2197
\]

(h) We define the covariance of joint random variables as
\[ \text{Cov}(X, Y) = E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y) \]

Calculate the covariance of \(X\) and \(Y\).

Finding \(E(XY)\),

\[
E(XY) = \int_0^1 \int_0^1 xy f_{xy}(x, y) dydx = 3 \int_0^1 \int_0^1 (x^2 y^3 + x^3 y^2) dydx =
\]

\[
= 3 \int_0^1 \left[ \frac{x^2}{4} + \frac{x^3}{3} \right] dx = 3 \left[ \frac{1}{4} + \frac{1}{3} \right] = \frac{1}{2}
\]

\[ \text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0.50000 - \left( \frac{17}{24} \right)^2 = -0.00174 \]

(h) Are \(X\) and \(Y\) independent?

Answer: NO

There are two ways to see this: (1) Their joint PDF function is not equal to the product of the marginal PDFs of \(X\) and \(Y\), or (2) their covariance is not zero.

**Problem 2** (36 points)

Let \(X\) be a random variable with the following CDF:

\[
F_X(x) = \begin{cases} 
0 & \text{for } x < 0 \\
x & \text{for } 0 \leq x < \frac{1}{4} \\
x + \frac{1}{2} & \text{for } \frac{1}{4} \leq x < \frac{1}{2} \\
1 & \text{for } x \geq \frac{1}{2}
\end{cases}
\]

(a) Plot \(F_X(x)\) and explain why \(X\) is a mixed random variable.

\(X\) is both a continuous random variable and discrete random variable (hence, a mixed random variable) because of the discontinuity in the shape of \(F_X(x)\). Plot appears below:
(b) Find \( P(X \leq 1/3) \).

\[
P(X \leq \frac{1}{3}) = F_X \left( \frac{1}{3} \right) = x + \frac{1}{2} = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}
\]

(c) Find \( P(X \geq 1/4) \).

\[
P(X \geq \frac{1}{4}) = 1 - P(X < \frac{1}{4}) = 1 - P(X < \frac{1}{4}) + P(X = \frac{1}{4})
\]

\[
= 1 - F_X \left( \frac{1}{4} \right) + \frac{1}{2} = 1 - \frac{3}{4} + \frac{1}{2} = \frac{3}{4}
\]
(d) Write the CDF of $X$ in the form, $F_X(x) = C_X(x) + D_X(x)$, where $C_X(x)$ is a continuous function and $D_X(x)$ is a discrete (staircase) function, for example,

$$D_X(x) = \sum_k a_k u(x - x_k)$$

We seek the form, $F_X(x) = C(x) + D(x)$; thus,

$$C_X(x) = \begin{cases} 
0 & \text{for } x < 0 \\
x & \text{for } 0 \leq x < \frac{1}{2} \\
\frac{1}{2} & \text{for } x \geq \frac{1}{2}
\end{cases}$$

and

$$D_X(x) = \begin{cases} 
0 & \text{for } x < \frac{1}{4} \\
\frac{1}{2} & \text{for } x \geq \frac{1}{4}
\end{cases}$$

or $$D_X(x) = \frac{1}{2} u(x - \frac{1}{4})$$

where $u(x)$ is the unit step function.

(e) Find continuous CDF $c(x)$ from $C_X(x)$ in part (d).

Using the relationship that $c(x)$ is the derivative of $C_X(x)$.

$$c(x) = \begin{cases} 
0 & \text{for } x < 0 \text{ and } x \geq \frac{1}{2} \\
1 & \text{for } 0 \leq x < \frac{1}{2}
\end{cases}$$

(f) Find $E(X)$ using

$$E(X) = \int x c(x) \, dx + \sum_k x_k a_k$$

$$E(X) = \int_0^{\frac{1}{2}} x \, dx + \sum_{k=\frac{1}{4}}^{\frac{1}{2}} \frac{1}{4} \left( \frac{1}{2} \right) = \frac{1}{2} \left( \frac{1}{2} \right)^2 + \frac{1}{4} \left( \frac{1}{2} \right) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$
Problem 3 (16 points)

The covariance can be computed for a list of data using the equation

\[\text{Cov}(X, Y) = \frac{\sum_{i=1}^{n} (X_i - \mu_X)(Y_i - \mu_Y)}{n - 1}\]

Given the data for random variables \(X\) and \(Y\), calculate the covariance.

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<th>(Y_i)</th>
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<th>((Y_i - \mu_Y))</th>
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Calculate the value of the covariance.

Solution:

Calculating the mean value \(\mu_X\) of R. V. \(X\) is found from

\[\mu_X = \frac{\sum X_i}{n} = \frac{44}{9} = 4.89\quad \text{and}\quad \mu_Y = \frac{\sum Y_i}{n} = \frac{49}{9} = 5.44\]

Next, we calculate the \((X_i - \mu_X)\) and \((Y_i - \mu_Y)\) quantities and then take their product as illustrated in the table below.
The covariance is given by

\[
Cov(XY) = \frac{\sum (X_i - \mu_X)(Y_i - \mu_Y)}{n-1} = \frac{-64.57}{9-1} = -8.07
\]

https://www.wikihow.com/Calculate-Covariance