Problem 1 (8 points)

How many ways can you select three cards from a group of seven non-identical cards?

Answer:

\[
\binom{n}{k} = \binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7!}{3!(4!)} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = \frac{7 \times 5}{1} = 35
\]

Problem 2 (12 points)

How many ways are there of throwing two indistinguishable dice? The answer is 21. Explain why.

Answer:

There are 6 ways of throwing the first die, and 6 ways of throwing the second die, and since the die throws are independent, there are \(6 \times 6 = 36\) possibilities. But a 2 and a 3 is the same as a 3 and a 2, and so on, if the dice are indistinguishable. Therefore, there are less than 36 ways of throwing two dice because of this fact. Of the 36 possibilities when the dice are distinguishable, 6 are pairs of one number (1 plus 1, 2 plus 3, etc.). That leaves 30 possibilities where the dice have different numbers; there are \(30/2 = 15\) possible results when we do not distinguish one die from the other die. These 15, plus the 6 results with identical numbers on each die, give 21 ways.

An alternate argument to show that 21 is the right answer is with the matrix of all 36 possibilities when die order matters.
The diagonal contains six ways with identical numbers for the two dice. There are 15 pairs of numbers in the upper diagonal and 15 pairs of numbers in the lower diagonal. But when order is not important the 15 pairs of numbers are equivalent to each other. So only 15 of the pairs of numbers are needed to meet the condition of the two dice being indistinguishable – we add the 6 ways with identical numbers to the 15 pairs which equals 21 ways.

**Problem 3** (8 points)

Given random variable $X$ with the probability for an event given by

$$P_X(x) = \frac{1}{3x} \quad \text{for } x = 1, 2, 3, 4$$

where $x$ only takes on values of 1, 2, 3, 4 for this event. Find the expected value of the event.

**Answer:**

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<td>$1/6$</td>
<td>$1/9$</td>
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Expectation value $= \sum_{1,2,3,4} xP_X(x) = 1 \times (1/3) + 2 \times (1/6) + 3 \times (1/9) + 4 \times (1/12) = 4/3$
Problem 4 (8 points)

A company makes electronic gadgets. One out of every fifty gadgets are faulty, but the company doesn't know which units are faulty until a buyer complains. Suppose the company makes a $3 profit on the sale of each working gadget, but losses $80 for every faulty gadget because they must repair the unit and return it to the customer. What is the long-term expected profit or loss per unit?

Answer:
The expectation value can be used to estimate the average profit or loss of each unit produced.

\[
EX = \frac{49}{50} \times 3.00 + \frac{1}{50} \times (-80.00) = \frac{147.00}{50} - \frac{80.00}{50} = \frac{67.00}{50}
\]

Dollars per unit = $1.34/unit

Since the expected value is positive, the company can expect to make a profit. On average, they make a profit of $1.34 per gadget produced.

Problem 5 (10 points)

The figure below shows three normal distributions labelled \(n1\), \(n2\), and \(n3\). Answer the flowing questions.

(a) Which one is the standard normal? What is its mean and variance?

(b) Which curve has the smallest standard deviation?

(c) Which curve has the smallest mean?

(d) For which curve is the probability \(P(X > 0)\) the biggest?

(e) For which curve is probability \(P(X > 0)\) the smallest?
Answers:
(a) Distribution \( n3 \) is the standard normal curve. It is the curve for \( Z \sim N(0,1) \), the mean is 0 and the variance is 1 (the standard deviation is 1 as well).

(b) Distribution \( n1 \), has the smallest standard deviation.

(c) Distributions \( n1 \) and \( n2 \) seem to have the same mean; both are smaller than \( n3 \).

(d) Distribution \( n3 \)

(e) Distribution \( n1 \)

Problem 6 (10 points)

Starting with \( \text{Var}(aX + b) = E[(aX + b)^2] \), where \( a \) and \( b \) are constants, and \( X \) is a continuous random variable; prove that

\[
\text{Var}(Y) = a^2 \text{Var}(X),
\]

given that random variable \( Y = aX + b \).
Proof: (Note: The most important point to be made in your solution is show how the constant b cancels in the derivation.)

With \( Y = aX + b \), then \( E[Y] = aE[X] + b = a\mu_X + b \). We can write

\[
\text{Var}[Y] = E\left[(Y - E[Y])^2\right] = E\left[(aX + b - aE[X] - b)^2\right] \quad b \text{ and } -b \text{ cancel}
\]

\[
= E\left[(aX - aE[X])^2\right] = E\left[a^2(X - E[X])^2\right] \quad \mu_X = E[X]
\]

\[
= a^2E\left[(X - \mu_X)^2\right] = a^2 \text{Var}[X] \quad \Leftarrow
\]

Alternate derivation:

\[
\text{Var}[Y] = E\left[Y^2\right] - (E[Y])^2 = E\left[(aX + b)^2\right] - (aE[X] + b)^2
\]

\[
= E\left[a^2X^2 + 2abX + b^2\right] - a^2(E[X])^2 - 2abE[X] - b^2
\]

\[
= a^2E\left[X^2\right] + 2abE[X] + b^2 - a^2(E[X])^2 - 2abE[X] - b^2
\]

\[
= a^2\left(E\left[X^2\right] - (E[X])^2\right) = a^2\text{Var}[X] \quad \Leftarrow
\]

because \( \text{Var}[X] = \left(E\left[X^2\right] - (E[X])^2\right) \)

Problem 7 (8 points)

Let \( X \) be a continuous random variable with PDF defined by

\[
f_X(x) = \begin{cases} 
2x & \text{for } 0 < x \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

Find the probability \( P\left(\frac{1}{3} \leq x \leq \frac{2}{3}\right) \).

Answer:

We integrate the PDF of this random variable between the limits of \( x = 1/3 \) and \( 2/3 \).
In probability we introduced the cumulative distribution function (CDF) for random variable $X$. What is the meaning or interpretation of the CDF?

**Answer:**

The cumulative distribution function (CDF) is the probability that the variable takes a value less than or equal to $x$. In symbols we can express it as

$$F_X(x) = P[X \leq x] \quad \text{for all } x \in \mathbb{R}$$

*From Section 3.2.1 in the textbook.*

**Problem 9 (16 points)**

Given the probability density function $f_X(x)$ as plotted below.

(a) Determine the constant $a$ as used in the plot as drawn.
Solution:

We use the fact that the area under the PDF $f_X(x)$ must be unity. Writing the expression for the area gives an equation we can solve for constant $a$. We break the figure into four segments each of width $\frac{1}{4}$.

\[
\text{Area} = \left[ \frac{1}{4} \left( \frac{a}{2} \right) + \frac{1}{4} (a) + \frac{1}{4} \left( \frac{a}{2} \right) + \frac{1}{4} (2a) \right] = \frac{a}{8} + \frac{a}{4} + \frac{a}{8} + \frac{a}{2} = 1
\]

Solving, $a = 1 \iff$

(b) Sketch the cumulative distribution function $F_X(x)$ corresponding to the PDF in part (a) above. I am only looking for the general behavior of $F_X(x)$.

[Note: The extra credit part of this examination asks for a detailed sketch with the break points quantified.]
**Solution:**

Consider a computer system with Poisson job-arrival stream at an average of 2 per minute (parameter $\lambda$). Determine the probability that in any one-minute interval that 0 jobs arrive. [Note: $e^1 = 2.7183; e^2 = 7.3891; e^3 = 20.0855; \text{etc.}]

**Problem 10** (6 points)

No job arrivals: ($\lambda = 2 \text{ jobs/min and } k = 0$)

\[ P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!} = e^{-2} \frac{2^0}{0!} = e^{-2} = 0.1353 \leq \]

**Problem 11** (6 points)

Past records show that the times (in seconds) taken to run 100 meters by children at a school can be modelled by a Gaussian distribution with mean $\mu = 16.1$ seconds and a standard deviation $\sigma = 1.6$ seconds. On sports day the school awards certificates to the fastest 30% of the children in the school. Estimate the slowest time taken to run 100 meters for which a child will be awarded a certificate. [Refer to the standard normal distribution table at the end of the examination.]
Answer:

We have \( \mu = 16.1 \) seconds and \( \sigma = 1.6 \) seconds (Variance = 2.56 seconds\(^2\)). We use the table to estimate the value of \( z_1 \) corresponding to 30% of the children. From the table we find that 0.300 equates to a value of \( z_1 = -0.525 \). Now using the relationship

\[
    z_1 = \frac{-x + \mu}{\sigma} \quad \Rightarrow \quad 0.525 = \frac{-x + 16.1}{1.6}; \quad \therefore \quad x = 16.1 - 0.525(1.6) = 15.26 \text{ sec}
\]

Hence, children running faster than 15.26 seconds receive a certificate.

EXTRA CREDIT: (up to 15 points)

The extra credit problem is a continuation of Problem 9. Draw the cumulative distribution function \( F_X(x) \) corresponding to the PDF in part (a) of problem 9. Quantify the break points at \( x = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \) and 1, thus, allowing you to accurately draw the CDF.
Solution:

\[ f_X(x) = \begin{cases} 
4x & \text{for } 0 \leq x < \frac{1}{4} \\
1 & \text{for } \frac{1}{4} \leq x < \frac{1}{2} \\
\frac{1}{2} & \text{for } \frac{1}{2} \leq x < \frac{3}{4} \\
2 & \text{for } \frac{3}{4} \leq x < 1 
\end{cases} \]

\[ F_X(x) = \begin{cases} 
2x^2 & \text{for } 0 \leq x < \frac{1}{4} \text{ (parabolic)} \\
x & \text{for } \frac{1}{4} \leq x < \frac{1}{2} \\
x/2 & \text{for } \frac{1}{2} \leq x < \frac{3}{4} \\
2x & \text{for } \frac{3}{4} \leq x < 1 
\end{cases} \]

0 at \( x = 0 \), 1/8 at \( x = \frac{1}{4} \), 3/8 at \( x = \frac{1}{2} \), 1/2 at \( x = \frac{3}{4} \), and 1 at \( x = 1 \); and unity for \( x > 1 \).

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