Problem 1 Channel Capacity and Nyquist Bandwidth (10 points)

(a) Claude Shannon (at Bell Telephone Laboratories) discovered an equation that gives the highest possible channel capacity of a communication system that can be achieved in the presence of noise (white Gaussian noise to be specific). The equation is

\[
C = B \cdot \log_2 (1 + SNR)
\]

where \(B\) is the channel bandwidth (in Hz) and \(SNR\) is signal-to-noise as a power ratio. Suppose a communication system utilizes a frequency band from 3 MHz to 4 MHz and requires a \(SNR\) (in dB) of at least 24 dB to operate without unacceptable bit errors. Calculate the maximum channel transmission capacity \(C\) as predicted by the Shannon equation. [Refer to Section 1.5 of Agbo & Sadiku, page 10.]

[Hint: \(SNR_{dB} = 10 \log_{10}(SNR)\) when stated in dB, so be careful about mixing the two logarithmic bases in this problem. The \(SNR\) quantity in the Shannon capacity equation is numerical (not dB) and the logarithm is base 2.]

**Answer:** We know that the bandwidth \(B = 4\) MHz \(- 3\) MHz = 1 MHz, and a \(SNR\) of 24 dB is numerically equal to 251 because \(SNR_{dB} = 24\) dB = \(10 \log_{10}(SNR)\) so that \(SNR = 10^{2.4} = 251\). Plugging these numbers into Shannon’s equation gives

\[
C = 1\, \text{MHz} \times \log_2 (1 + 251) = 10^6 \times \log_2 (252) = 10^6 \times \frac{\log_{10} (252)}{\log_{10} (2)} \approx 8 \, \text{Mbps}
\]

(b) Next, we consider the Nyquist formula which tells us the digital channel capacity as a function of the number of levels per symbol. The smallest number of levels per symbol is binary which is two levels. Other forms of coding can have more than two levels per symbol. Whereas Shannon’s equation tells us the maximum data rate possible in the presence of noise, Nyquist’s equation tells us the data rate \(C\) as a function of bandwidth \(B\) and the number of signal levels per symbol \(M\) we can achieve. Nyquist’s equation is

\[
\text{Nyquist channel capacity } C = 2B \cdot \log_2 (M) \quad \text{(in bits/second)}
\]
where $M$ is the number of signal levels per symbol. Calculate the Nyquist data rate given the same bandwidth as in part (a) above and assume (a) 2 levels per symbol and also (b) 8 signal levels per symbol. Compare the two cases for capacity $C$.

**Answer:** The bandwidth is now $2B = 2$ MHz and $\log_2(2) = 1$ for two levels per symbol and $\log_2(8) = 3$ for eight level per symbol because $2^3 = 8$. Substituting these values into the Nyquist equation gives

(a) Nyquist channel capacity $C = 2 \times 10^6 \cdot \log_2(2) = 2 \times 10^6 \times 1 = 2$ Mbps

(b) Nyquist channel capacity $C = 2 \times 10^6 \cdot \log_2(8) = 2 \times 10^6 \times 3 = 6$ Mbps

Thus, the channel capacity is tripled for 8 levels/symbol compared to 2 levels/symbol.

**Problem 2 Square Law Device (20 points)**

Some components in communication systems must be nonlinear as opposed to linear time-invariant components. A square-law device (such as a MOSFET transistor operated in its saturation region) is a nonlinear component which is useful for signal detection and mixing in communication systems. You are given a square-law component with an input to output relationship of

$$y(t) = A + B(g(t))^2$$

where $g(t)$ is the input, $A$ and $B$ are constants and $y(t)$ is the output.

(a) To explore the behavior of this device we let the input signal $g(t)$ be a sinusoidal tone, that is, $g(t) = \cos(2\pi f t)$. Write the expression for $y(t)$ and use the trigonometric identity to convert the cosine-squared term to a first power cosine term. The input frequency is $f$, what is the frequency of the output $y(t)$?

**Answer:** The square-law device generates a frequency that is the double of the single tone frequency $f$. To show this we make use of the trigonometric identity: $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$.

$$y(t) = A + B(g(t))^2 = A + B[\cos(2\pi ft)]^2 = A + \frac{B}{2}[1 + \cos(2 \cdot 2\pi ft)]$$

$$y(t) = \left(A + \frac{B}{2}\right) \cdot \cos(2 \cdot 2\pi ft) = \left(A + \frac{B}{2}\right) \cdot \cos(2\omega t), \text{ where } 2\pi f = \omega$$

(b) Unfortunately, no actual device has exactly a square-law characteristic. Some devices are close, but not perfectly square-law. A better way to express the input-output characteristic is with a power series written in the format,

$$y(t) = A + Bg(t) + C(g(t))^2 + D(g(t))^3 + \text{ other terms}.$$
The first two terms of the expression are linear and can't generate new frequencies when driven by a single tone sinusoidal input. In part (a) you found that the third term does generate an additional frequency. The fourth term, although generally small in magnitude, is also nonlinear. What frequencies does the cubic term (that is, \( D[g(t)]^3 \)) generate when driven by \( g(t) = \cos(2\pi f) \)?

**Answer:** The cubic term in the series generates a frequency that is triple of the frequency of \( g(t) \), that is, frequency \( 3f \). This comes from using the trigonometric identity of \( \cos^3(x) = \frac{1}{4}[3\cos(x) + \cos(3x)] \). Thus, the \( \cos^3(2\pi f) \) term gives us both a \( \cos(2\pi f) \) term (not so interesting) and a \( \cos(3\cdot2\pi f) \) term (which is a new frequency being introduced).

**Problem 3 RC Low-Pass Filter Problem** (20 points)

Filters are a very important part of communication systems. Before we discuss filters in more detail in class, this problem is assigned to start thinking about filters; we select a very simple filter, namely, the RC low-pass filter. This is illustrated in the figure below:

(a) The transfer function of a filter is defined as the ratio of output signal to input signal assuming a sinusoidal excitation at the input \((i.e., g(\omega))\). The transfer function is

\[
H(\omega) = \frac{y(\omega)}{g(\omega)}.
\]

where \( g(\omega) = e^{j\omega t} \) and \( y(\omega) = H(\omega) \cdot e^{j\omega t} \). Solve for \( H(\omega) \) and find the -3 dB bandwidth (half power) for this low-pass filter. [Note: This should be review from your EE 400 class.]

**Answer:** The Transfer function is

\[
H(\omega) = \frac{(1/ j2\pi fC)}{R + (1/ j2\pi fC)} = \frac{1}{1 + j2\pi fRC} \Leftrightarrow \\
\text{and } f_{-3dB} = \frac{1}{2\pi RC} \text{ is the -3 dB bandwidth.}
(b) Next, we want to explore both the unit step and impulse responses of the RC low-pass filter. A unit step $u(t)$ input is shown in the figure below:

![Unit step function](image)

If we apply the unit step $u(t)$ to the input of the $RC$ low-pass filter what is the output waveform? Sketch it on the graph below.

**Answer:** The response $y_u(t) = (1 - e^{-t/RC})u(t)$.

![Output waveform](image)

(c) The unit impulse response $h(t)$ to a unit impulse function (i.e., Dirac delta function $\delta(t)$) is an important function in the characterization of a network (e.g., filter). Determine the unit impulse response $h(t)$ for this $RC$ low-pass filter and sketch it on the graph below.

**Answer:** The unit impulse response is

$$h(t) = \frac{1}{RC} e^{-t/RC}u(t).$$
(d) How are the unit step response and the unit impulse response related?

**Answer:** Differentiation of the unit step response yields the unit impulse response.

\[
\frac{du(t)}{dt} = \delta(t) \quad \text{and} \quad u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau
\]

leading to \( \frac{dy_u(t)}{dt} = h(t) \quad \text{and} \quad y_u(t) = \int_{-\infty}^{t} h(\tau) d\tau \)

Thus, \( h(t) = \frac{d}{dx} [y_u(t)] = \frac{d}{dx} \left[ 1 - e^{-t/RC} \right] = \frac{1}{RC} e^{-t/RC} \).

**Problem 4 Free-Space Radio Propagation** (30 points)

This problem involves free-space radio wave propagation using antennas between transmitter and receiver assuming line of sight transmission. In EE442 we do not study antennas, but rather use a few simple results from antenna theory.

Path loss PL(dB) is the attenuation of a signal traveling from transmitter to a receiver along a free path. Let power \( P_t \) be the power emitted by the transmitter’s antenna and power \( P_r \) be the signal power received by the receiver’s antenna. Then the path loss is

\[
PL(dB) = 10 \cdot \log_{10} \left( \frac{P_r}{P_t} \right) = -10 \cdot \log_{10} \left( \frac{\lambda^2}{(4\pi)^2 d^2} \right) \quad \text{(note the minus sign)}
\]

where \( \lambda = \) wavelength of signal and \( d = \) distance between the transmitter and the receiver antennas.

This equation comes from the Friis free-space model and we have assumed the antennas gain are both unity (this not important unless you know antenna theory). Your cell phone transmits about \( \frac{1}{2} \) watt of power when it is communicating with the base station in your cellular network. Assume the frequency of transmission in the uplink band to be 915 MHz.

(a) What is the transmit power of the cell phone in dBm?

\[
P_t[\text{dBm}] = 10 \cdot \log_{10} \left( \frac{P_t[\text{mW}]}{1 \text{ mW}} \right) = 10 \cdot \log_{10} \left( \frac{500 \text{ mW}}{1 \text{ mW}} \right) = 27 \text{ dBm} \quad \Leftarrow
\]
(b) Suppose the distance $d$ between your cell phone and the base station is one kilometer ($1 \text{ km} = 1000 \text{ m} = 3280 \text{ feet}$). What is the received power in dBm?

First we find the wavelength, 
$$\lambda = \frac{c}{f} = \frac{3 \times 10^8 \text{ m/SEC}}{9.15 \times 10^6 \text{ 1/SEC}} = 0.328 \text{ m}.$$  

From the Friis equation for free-space propagation we can write

$$10 \cdot \log_{10} \left( \frac{P_t}{P_r} \right) = -10 \cdot \log_{10} \left( \frac{P_t}{P_f} \right); \quad \text{Thus,} \quad \frac{P_r}{P_t} = \frac{\lambda^2}{(4\pi)^2 d^2} \quad \text{so} \quad P_r = \frac{P_t \lambda^2}{(4\pi)^2 d^2}$$

$$P_r = \frac{(500 \text{ mW})(0.328 \text{ m})^2}{(4\pi)^2 (1000 \text{ m})^2} = \frac{500 \times 0.1075}{(12.57)^2 \times 10^6} \text{ mW} = 3.404 \times 10^{-7} \text{ mW} \quad \Leftarrow$$

In dBm we get

$$P_r[\text{dBm}] = 10 \cdot \log_{10} \left( \frac{3.404 \times 10^{-7} \text{ mW}}{1 \text{ mW}} \right) = -64.7 \text{ dBm} \quad \Leftarrow$$

(c) Suppose the weakest signal the base station can still pick up is -81 dBm. What is the greatest distance $d$ that your cell phone can be from the base station and still operate?

$$-81 \text{ dBm} = 10 \cdot \log_{10} \left( \frac{P_r \text{[mW]}}{1 \text{ mW}} \right) \quad \Rightarrow \quad 10^{-8.1} = 10^{\log(P_r)}$$

$$P_r = 7.943 \times 10^{-9} \text{ mW} = \frac{500 \times 0.1075}{(12.57)^2 d^2} \text{ mW}; \quad \text{Solve for} \quad d$$

$$d^2 = \frac{500 \times 0.1075}{(12.57)^2 P_r} \text{ m}^2 = \frac{500 \times 0.1075}{(157.9)(7.943 \times 10^{-9})} \text{ m}^2$$

$$d = 6,546 \text{ m} \quad \Leftarrow \quad \text{(Note: This is about 4 miles.)}$$

**Problem 5 Channel Capacity in Limit of Infinite Bandwidth**  \hspace{1cm} (20 points)

All communication systems are limited bandwidth systems. In Problem 1 above we considered the channel capacity of a communication system in the presence of noise (white Gaussian noise to be specific). The Shannon channel capacity equation is

$$\text{Shannon channel capacity} \quad C = B \cdot \log_2 \left( 1 + \frac{S}{N} \right) \quad \text{(units: bits/second)}$$

where $B$ is the channel bandwidth (in Hz), $S$ is the signal power (in watts) and $N$ is the total noise power (again in watts). The ratio of $(S/N)$ is called the signal to noise ratio and is an important communication system parameter. Unfortunately, noise is always
present in all communication channels, so we must account for it in any realistic communication system.

In this problem we are going to play the “what if” game by asking what would happen to the channel capacity if the channel bandwidth $B$ were to become infinite. For such a case, if there were no noise power, the channel capacity would obviously become infinite as the bandwidth approaches infinity (obviously not physically possible in the real world). So, this will not happen, but now consider the presence of noise.

Now for the problem: A common noise model in use is the “additive white Gaussian noise” (AWGN) model. For AWGN the noise power density $N_n$ is constant for all frequencies with no upper limit in frequency. The physical units for $N_n$ is noise power (in watts) per Hertz. Total channel noise power $N$ in the Shannon equation is given by $N = N_n \cdot B$; hence, $N$ is proportional to bandwidth $B$. As $B$ approaches infinity, so does the total channel noise power $N$. Furthermore, if the bandwidth $B$ goes to infinity, so would the signal power $S$ if we transmit fully utilizing the available (infinite) bandwidth.

Now start with Shannon’s channel capacity equation and let bandwidth $B$ go to infinity in the limit, that is,

$$
\lim_{B \to \infty} (C) = C_\infty = \lim_{B \to \infty} \left[ B \cdot \log_2 \left( 1 + \frac{S}{N_n B} \right) \right]
$$

Find an expression for $C_\infty$ in terms of $(S/N_n)$.

$$
\left[ \text{Hint: } \lim_{x \to \infty} \left( x \cdot \log_2 \left[ 1 + \frac{1}{x} \right] \right) = \log_2 (e) = 1.443 \right]
$$

$$
C_\infty = \lim_{B \to \infty} \left( B \cdot \log_2 \left( 1 + \frac{S}{N_n B} \right) \right); \quad \text{Next we write this in the form}
$$

$$
C_\infty = \lim_{B \to \infty} \frac{S}{N_n} \left( \frac{N_n B}{S} \cdot \log_2 \left( 1 + \frac{S}{N_n B} \right) \right)
$$

Make the substitution of $x = \frac{N_n B}{S}$; then we can write $C_\infty$ as

$$
C_\infty = \lim_{x \to \infty} \frac{S}{N_n} \left( x \cdot \log_2 \left( 1 + \frac{1}{x} \right) \right) = 1.443 \cdot \frac{S}{N_n}
$$

Thus, Even if infinite bandwidth were possible, $C$ would be finite.

Added note: $\left( \text{Note: } \log_a z = \frac{\ln z}{\ln a} \text{ is useful if } a = 2 \text{ and } \ln \text{ stands for the natural logarithm (base } e). \right)$