Problem 1 Phasor Representation of AM (20 points)

In lecture we discussed the use of phasors to represent amplitude modulated signals where the message signal is a tone (that is, a single frequency sinusoid). We found the AM time domain signal could be written in the form,

\[ \phi_{\text{AM}}(t) = \text{Re} \left[ e^{j\omega_c t} \left( 1 + \frac{e^{j\omega_m t}}{2} + \frac{e^{-j\omega_m t}}{2} \right) \right] \]

where the carrier frequency is \( \omega_c \) and the tone signal frequency are \( \omega_m \).

Using trigonometric identities, show that this expression for \( \phi_{\text{AM}}(t) \) contains terms of \( \cos(\omega_c t) \), \( \cos(\omega_c t + \omega_m t) \) and \( \cos(\omega_c t - \omega_m t) \). Show your work and identify the trigonometric identities you used in your calculation.

Answer: (Trigonometric identities in black)

We use identities \( \cos(\omega_m t) = \frac{1}{2} \left( e^{j\omega_m t} + e^{-j\omega_m t} \right) \) & \( e^{j\omega_c t} = \cos(\omega_c t) + j \sin(\omega_c t) \); substituting,

\[ \phi_{\text{AM}}(t) = \text{Re} \left[ (\cos(\omega_c t) + j \sin(\omega_c t)) \times \left( 1 + \cos(\omega_m t) \right) \right] \]

And \( \cos(\omega_c t) \cdot \cos(\omega_m t) = \frac{1}{2} \left( \cos((\omega_c - \omega_m) t) + \cos((\omega_c + \omega_m) t) \right) \)

\[ \phi_{\text{AM}}(t) = \text{Re} \left[ \cos(\omega_c t) + \frac{1}{2} \left( \cos((\omega_c - \omega_m) t) + \cos((\omega_c + \omega_m) t) \right) + j(\cdots) \right] \]

Thus,

\[ \phi_{\text{AM}}(t) = \cos(\omega_c t) + \frac{1}{2} \left( \cos((\omega_c - \omega_m) t) + \cos((\omega_c + \omega_m) t) \right) \]
Problem 2  AM Modulation Index & Power Efficiency (20 points)

The waveform shown below is a tone modulated AM signal as might be displayed on an oscilloscope. Both the carrier and the message (tone) are sinusoidal. Find the modulation index $\mu$ and power efficiency $\eta$ of the AM signal.

**Answer:**
The modulation index $\mu$ is defined as the ratio of the peak (maximum) value of the message signal, denoted by $m_p$, to the peak amplitude of the carrier signal, denoted by $A_C$. From the drawing we see the modulation from the tone sinusoid has a minimum value of 1 volt and maximum value of 4 volts; that is a peak-to-peak excursion of 3 volts, thus, $m_p = 1.5$ volts. For the carrier wave the amplitude $A_C$ is 2.5 volts.

$$\mu = \frac{m_p}{A_C} = \frac{1.5 \text{ volts}}{2.5 \text{ volts}} = 0.6$$

For the power efficiency $\eta$ is computed from

$$\eta = \frac{\mu^2}{2 + \mu^2} = \frac{(0.6)^2}{2 + (0.6)^2} = \frac{0.36}{2.36} = 0.153 \text{ or } 15.3\%$$
Problem 3 Nonlinear AM Modulator (30 points)

Given the nonlinear AM modulator shown below:

The total diode input voltage $v_D$ is smaller than the turn-on voltage of the diode and the diode current is

$$i_D = \beta (4v_D + v_D^2)$$

Note this is a linear term plus a quadratic term in voltage $v_D$. The carrier is $A_C \cos(\omega_c t)$ and the message signal is a DC term $\alpha$ plus the time-varying message $m(t)$. The message signal bandwidth is $B$ Hz, where $\omega_c >> 2\pi B$. Derive an expression for the output signal $y(t)$ given that the center of the band-pass filter is at $\omega_c$. [Note: Assume all of current $i_D$ flows through resistor $R$ (and $R = 1\, \Omega$).]

Solution:

$$x(t) = i_D R = \beta \left(4v_D + v_D^2\right) \cdot R, \quad \text{but } R = 1\, \Omega$$

Substituting for $v_D (= \alpha + m(t) + A_c \cos(\omega_c t))$,

$$x(t) = 4\beta \left[ (\alpha + m(t) + A_c \cos(\omega_c t)) \right] + \beta \left[ (\alpha + m(t) + A_c \cos(\omega_c t))^2 \right]$$

$$x(t) = 4\beta \left[ (\alpha + m(t) + A_c \cos(\omega_c t)) \right] + \beta \left[ (\alpha + m(t))^2 + 2A_c (\alpha + m(t) \cdot \cos(\omega_c t)) + A_c^2 \cos^2(\omega_c t) \right]$$

But we know $A_c^2 \cos^2(\omega_c t) = \frac{A_c^2}{2} (1 + \cos(2\omega_c t))$
\[ x(t) = 4\beta\left[ (\alpha + m(t) + A_c \cos(\omega_c t)) \right] + \beta\left[ (\alpha + m(t))^2 + 2A_c (\alpha + m(t) \cdot \cos(\omega_c t)) + A_c^2 \cos^2(\omega_c t) \right] \]
\[ x(t) = \left[ 4\beta(\alpha + m(t)) + \beta\left( (\alpha + m(t))^2 + \frac{\beta A_c^2}{2} \right) + \left( 2\beta A_c (\alpha + m(t)) \cdot \cos(\omega_c t) \right) + \frac{\beta A_c^2}{2} \cos(2\omega_c t) \right] \]

The band-pass filter (BPF) passes only terms of \( \cos(\omega_c t) \), thus \( y(t) \) is
\[ y(t) = 4\beta A_c \cdot \cos(\omega_c t) + 2\beta A_c (\alpha + m(t)) \cdot \cos(\omega_c t) \]
\[ y(t) = \left( 4\beta A_c + 2\beta A_c \alpha \right) \cdot \cos(\omega_c t) + m(t) \cdot \cos(\omega_c t) \]

**Problem 4  RLC Resonator** (30 points) [This should be review.]

One possible frequency selective circuit is a simple LRC resonator as schematically drawn below. The resonant frequency of the circuit is
\[ f_{\text{resonance}} = \frac{1}{\sqrt{LC}} \]

The Quality factor of a resonant circuit (Q-factor for short) is defined as the resonant frequency divided by the half-power (or -3 dB) bandwidth; in symbols it is
\[ Q = \frac{\text{resonance frequency}}{\text{half-power bandwidth}} = \frac{f_{\text{resonance}}}{B} \]

An RLC circuit finds use for selecting communication bands by tuning to the desired carrier frequency. It is also used for “demodulation” of amplitude modulation (AM) communication signals, “modulation” of frequency modulation (FM) communication signals and for establishing oscillation frequencies in local oscillators as used in superheterodyne receivers. Consider the parallel RLC circuit as shown below:
(a) Derive the transfer function $H(\omega)$ for this parallel RLC circuit. Assume the sinusoidal steady-state in deriving the transfer function. We define $H(\omega)$ as the ratio of the current $i_R$ flowing through the resistor divided by the input current $i(t)$.

**Answer:**

$$H(\omega) = \frac{i_R}{i(t)} = \frac{j\omega L}{R + j\omega L + (j\omega)^2 RCL}$$

(b) Derive the half-power bandwidth $B$ (i.e., the -3-dB bandwidth forming the frequency band between the -3 dB frequencies) for this circuit. Express bandwidth $B$ as a function of $R$, $L$ and $C$.

**Answer:**

$$\omega_1 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + (LC)^2} \quad \text{and} \quad \omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + (LC)^2}$$

Bandwidth $B = \omega_2 - \omega_1 = \frac{1}{RC}$

(c) Starting from the definition of Q-factor at the beginning of the problem statement; derive an expression for $Q$ as a function of the circuit parameters.

**Answer:**

Q-factor: $Q = \frac{\omega_0}{B} = \omega_0 RC = \frac{R}{\omega_0 L}$ \qquad \text{where} \quad \omega_0 = \frac{1}{\sqrt{LC}}$