**Problem 1 Double-sideband suppressed carrier signal** (30 points)

A carrier signal \(\cos(\omega_ct)\) is combined with a two-tone message signal \(m(t)\),

\[m(t) = 3\cos(\omega_m t) + \sin(3\omega_m t)\]

where \(\omega_m = 5\pi \times 10^3\) radians/sec and \(\omega_c = 2\pi \times 10^4\) radians/sec.

(a) Write an expression for the modulated signal \(m(t)\times \cos(\omega_c t) = \phi_{DSB}(t)\).

\[m(t) = 3\cos(\omega_m t) + \sin(3\omega_m t) \quad \text{and} \quad c(t) = \cos(\omega_c t)\]

\[\phi_{DSB}(t) = \left(3\cos(\omega_m t) + \sin(3\omega_m t)\right)\times \cos(\omega_c t)\]

\[\phi_{DSB}(t) = 3\cos(\omega_m t) \cdot \cos(\omega_c t) + \sin(3\omega_m t) \cdot \cos(\omega_c t)\]

We make use \(\cos \theta \cdot \cos \phi = \frac{1}{2} \left[ \cos(\theta - \phi) + \cos(\theta + \phi) \right]\)

\(\sin \theta \cdot \cos \phi = \frac{1}{2} \left[ \sin(\theta - \phi) + \sin(\theta + \phi) \right]\)

\[\phi_{DSB}(t) = \frac{3}{2} \left[ \cos ((\omega_c - \omega_m) t) + \cos ((\omega_c + \omega_m) t) \right] + \]

\[+ \frac{1}{2} \left[ \sin ((3\omega_m - \omega_c) t) + \sin ((3\omega_m + \omega_c) t) \right] \quad \Leftarrow\]

(b) At the receiver the modulated signal is received and input to the demodulator shown below.

\[\phi_{DSB}(t) \rightarrow x(t) \rightarrow \text{filter} \rightarrow y(t)\]

\[2 \cdot \cos(\omega_c t)\]
The carrier at the demodulator is $2 \cos(\omega_c t)$ and is followed by a filter. Find an expression for the signal $x(t)$.

\[
\phi_{DSB}(t) \times 2 \cos(\omega_c t) = x(t) = \\
\frac{3}{2} \left[ \cos((\omega_c - \omega_m) t) + \cos((\omega_c + \omega_m) t) \right] \times 2 \cos(\omega_c t) \\
+ \frac{1}{2} \left[ \sin((3\omega_m - \omega_c) t) + \sin((3\omega_m + \omega_c) t) \right] \times 2 \cos(\omega_c t)
\]

\[
x(t) = \frac{3}{2} \left( \cos(\omega_c - \omega_m - \omega_C) t + \cos(\omega_c - \omega_m + \omega_C) t \right) \\
+ \frac{1}{2} \left( \sin(3\omega_m - \omega_c - \omega_C) t + \sin(3\omega_m - \omega_c + \omega_C) t \right)
\]

\[
x(t) = \frac{3}{2} \left( 2 \cos(\omega_m t) + \cos(2\omega_c - \omega_m) t + \cos(2\omega_c + \omega_m) t \right) + \\
+ \frac{1}{2} \left( 2 \sin(3\omega_m t) + \sin(3\omega_m - 2\omega_c) t + \sin(3\omega_m + 2\omega_c) t \right)
\]

(c) Describe the filter parameters (or requirements) that you would select to recover the message signal $m(t)$ from $x(t)$?

The filter must pass frequencies $\omega_m$ and $3\omega_m$ radians/sec. But it must reject the other frequencies such as $(3\omega_m \pm \omega_C)$ and $(3\omega_m \pm 2\omega_C)$ and $(2\omega_C \pm \omega_m)$. That means the filter must pass frequencies $\omega_m = 5\pi \times 10^3$ radians/sec and $3\omega_m = 15\pi \times 10^3$ radians/sec. But it must reject $|3\omega_m \pm 2\omega_C| > 25\pi \times 10^3$ radians/sec. It would be a low-pass filter.

**Problem 2 Envelope Detector** (20 points)

We have an envelope detector to be used to cover the entire AM radio broadcast band from 550 kHz to 1600 kHz. The envelope detector consists of a smoothing capacitor in parallel with a resistor $R = 5 \, k\Omega$. The smoothing capacitor $C$ is tunable over the AM radio band. Our assignment
is to determine the range of smoothing capacitor values assuming that we must vary the capacitance value from one end of AM radio band to the other end of the AM radio band. For proper operation of the envelope detector the reciprocal of the $RC$ time constant must meet the criteria:

$$B_m << \frac{1}{RC} << f_C$$

Where $B_m$ is the bandwidth (or greatest frequency) of the baseband (voice) message $m(t)$ and $f_C$ is the carrier frequency of the radio station being considered. You decide to use the “geometric mean” of $B_m$ and $f_C$ to select the capacitance value for a particular station within the AM radio band. [Note: If you need to refer to what the “geometric mean” is, a good reference is https://en.wikipedia.org/wiki/Geometric_mean.] For AM radio each station is allowed $B_m = 5$ kHz. With $R = 5$ kΩ, determine the range of smoothing capacitor values to cover the entire AM radio band using the geometric mean to determine $RC$ for any AM radio station. In other words, one must choose a capacitor that will tune over what range of capacitance values to cover 550 kHz to 1600 kHz?

For each value of $f_C$ corresponding to the AM radio station’s carrier frequency, the geometric mean of the $RC$ value is determined using

$$RC = \frac{1}{\sqrt{B_m \cdot f_C}}; \text{ so capacitance } C = \frac{1}{R \sqrt{B_m f_C}}$$

Thus,

For $f_C = 550$ kHz and $B_m = 5$ kHz,

$$C = \frac{1}{R \sqrt{B_m f_C}} = \frac{1}{5000 \sqrt{5000 \times 550,000}} = 3.81 \times 10^{-9} \text{ F} = 3.81 \text{ nF} \Leftarrow$$

For $f_C = 1600$ kHz and $B_m = 5$ kHz,

$$C = \frac{1}{R \sqrt{B_m f_C}} = \frac{1}{5000 \sqrt{5000 \times 1,600,000}} = 2.24 \times 10^{-9} \text{ F} = 2.24 \text{ nF} \Leftarrow$$
Problem 3  Single-Tone Modulated AM  (15 points)

A single-tone message signal $m(t)$ is combined with a carrier signal to give an amplitude modulated signal of the form, 

$$\phi_{AM}(t) = A_C \left( 1 + A_m \cos(\omega_m t) \right) \times \cos(\omega_C t)$$

We are told that the maximum and minimum values of $\phi_{AM}(t)$ has values of 

$$|\phi_{AM}(t)|_{\text{maximum}} = A_C \left( 1 + A_m \right) = 1.0 \text{ volt}$$
$$|\phi_{AM}(t)|_{\text{minimum}} = A_C \left( 1 - A_m \right) = 0.4 \text{ volt}$$

The maximum value occurs when $\cos(\omega_m t) = 1$ and the minimum value occurs when $\cos(\omega_m t) = -1$. Determine the following parameters:

(a) The amplitude $A_C$ of the carrier signal.

Starting with $A_C + A_C A_m = 1.0$ and $A_C - A_C A_m = 0.4$

Solving these two simultaneous equations gives values for $A_C$ & $A_m$

Thus, $2A_C = 1.4$ yields $A_C = 0.7 \text{ volt}$  $\Leftarrow$

(b) The amplitude $A_m$ of the single-tone message signal.

Using this value for $A_C$ allows the calculation of $A_m$

$0.7 - 0.7A_m = 0.4$ yields $A_m = 0.429 \text{ volt}$  $\Leftarrow$

(c) The modulation index $\mu$.

The modulation index $\mu$ is given by

$$\mu = \frac{m_p}{A_C} = \frac{A_m}{A_C} = \frac{0.429}{0.7} = 0.61 \text{ (or 61%)}$$  $\Leftarrow$

Problem 4  Baseband Signal Recovery  (15 points)

A frequency-translated baseband signal $m(t)$ (frequency shifted by $\omega_c$) is given by $\phi(t) = m(t) \cdot \cos(\omega_c t)$. We may recover $m(t)$ by multiplying $\phi(t)$ by a local oscillator signal given by $\cos(\omega_c t + \theta)$. The parameter $\theta$ represents a phase shift. In this problem you are asked to investigate the effect of this offset in phase angle $\theta$. 
(a) The modulation product of $m(t)$ and $\cos(\omega_c t + \theta)$ is passed through a low-pass filter rejecting the double-frequency ($2\omega_c$) term. After filtering, what is the signal output?

\[
\phi(t) \cdot \cos(\omega_c t + \theta) = m(t) \cdot \cos(\omega_c t) \cdot \cos(\omega_c t + \theta) \\
= m(t) \left[ \frac{1}{2} \cos(\omega_c t - \omega_c t - \theta) + \frac{1}{2} \cos(2\omega_c t + \theta) \right] \\
= \frac{m(t)}{2} [\cos(-\theta) + \cos(2\omega_c t + \theta)]
\]

The filter removes the $\cos(2\omega_c t + \theta)$ term, therefore

\[
= \frac{m(t)}{2} \cos(-\theta)
\]

(b) Next, using the result from part (a) above, what is the filter’s output when $\theta$ is equal to $\pi/2$ radians?

\[
\frac{m(t)}{2} \cos(-\theta) \approx \frac{m(t)}{2} \cos\left(-\frac{\pi}{2}\right) = 0
\]

(c) How much phase shift $\theta$ can be tolerated for a decrease no greater than 5% of the magnitude at the filter’s output?

For a 5% decrease in response we require that $\cos(-\theta) = 0.95$, corresponding to $\theta = 18.2$ degrees (or 0.317 radian)

**Problem 5  Last Problem**  (20 points)

We have our usual double-sideband with carrier AM signal with single-tone message signal represented by

\[
\phi_{AM}(t) = (A_c + m(t)) \cdot \cos(\omega_c t) = (A_c + A_m \cos(\omega_m t)) \cdot \cos(\omega_c t)
\]

We know that the power in a pure sinusoidal signal given by $A \cdot \cos(\omega_c t)$ is simply $\text{Power} = \frac{1}{2}A^2$. That is, power $P$ is calculated from taking the mean square value of the signal:

\[
P_c = \left\langle A_c^2 \cos^2(\omega_c t) \right\rangle_{\text{average}} = \frac{A_c^2}{2}
\]
For an AM signal we define the total AM power $P_T$ to be the sum of the carrier power $P_C$ plus the sideband power $P_S$. Remember the sideband power is one-half the power in the single-tone modulating signal. Using this information derive an expression for the total power as a function of $P_C$ and the modulation index $\mu$. [Remember the modulation index is defined as the ratio of the maximum message (modulating) amplitude to the maximum carrier amplitude.]

The carrier power is calculated from

$$P_C = \langle A_C^2 \cdot \cos(\omega_c t) \rangle = \frac{A_C^2}{2},$$

and the sideband power is

$$P_S = \frac{1}{2} \langle m^2(t) \rangle = \frac{1}{2} \langle A_m^2 \cdot \cos^2(\omega_c t) \rangle = \frac{1}{2} \left( \frac{A_m^2}{2} \right)$$

$$P_T = P_C + P_S = \frac{A_C^2}{2} + \frac{1}{2} \left( \frac{A_m^2}{2} \right) = \frac{A_C^2}{2} \left[ 1 + \frac{1}{2} \left( \frac{A_m^2}{A_C^2} \right) \right]$$

Modulation index $\mu = \frac{\text{maximum message amplitude}}{\text{maximum carrier amplitude}} = \frac{A_m}{A_C}$

$$\therefore P_T = \frac{A_C^2}{2} \left[ 1 + \frac{\mu^2}{2} \right] = P_C \left[ 1 + \frac{\mu^2}{2} \right]$$