EE 442 Homework #3  
(Spring 2019 – Due February 18, 2019 )
Print out homework and do work on the printed pages.

**Problem 1 Phasor Representation of AM** (20 points)

In lecture we mentioned the use of phasors to represent amplitude modulated signals where the message signal is a tone (that is, a single frequency sinusoid). An amplitude modulated time domain signal can be written in the form,

$$\phi_{AM}(t) = \text{Re} \left[ e^{j\omega_c t} \left( 1 + \frac{e^{j\omega_m t}}{2} + \frac{e^{-j\omega_m t}}{2} \right) \right],$$

where the radian carrier frequency is $\omega_c$ and the tone modulating signal frequency is $\omega_m$. Using trigonometric identities, show that the expression for $\phi_{AM}(t)$ contains terms of $\cos(\omega_c t)$, $\cos(\omega_c t + \omega_m t)$ and $\cos(\omega_c t - \omega_m t)$. **Show your work in detail** and **identify the trigonometric identities** you used in your calculation.
Problem 2  Fourier Transform of Decaying Exponential Function  (20 points)

Consider the time-domain function
\[ f(t) = e^{-at} \]
where \( a \) is a fixed positive constant and \( t \) represents time.

(a) Find the Fourier transform \( F(\omega) \) of function \( f(t) \).

(b) Find the magnitude of \( F(\omega) \) (i.e., \( |F(\omega)| \)) and the phase angle \( \phi(\omega) \) of \( F(\omega) \).
(c) Sketch the general shape of $|F(\omega)|$ and the phase angle $\phi(\omega)$.

(d) If the function $f(t)$ is interpreted as the unit impulse response $h(t)$ of a network, given the form of $|F(\omega)|$ you found in parts (b) and (c) of this problem, what network do you think it is (you are already familiar with it)?

**Problem 3 Fourier Transform of Double-Sided Exponential Function** (20 points)

In the above problem we considered a single-sided decaying exponential. Now we have a double-sided decaying exponential time-domain function,

$$f(t) = e^{-|t|} = \begin{cases} e^{at} & \text{for } t < 0 \\ e^{-at} & \text{for } t > 0 \end{cases}$$
where \( a \) is a positive constant and the vertical bars represent the absolute value of \( t \).

(a) Find the Fourier transform \( F(\omega) \) of function \( f(t) \).

(b) Find the magnitude and phase of the expression you found in part (a) above.

(c) Compare the Fourier transform \( F(\omega) \) of this function \( f(t) \) to the Fourier transform of the one-side decaying exponential you found in Problem 2 above. What is unusual about the phase of the Fourier transform of the double-sided exponential? Is this realizable in a network or filter you could build in a laboratory? Explain your reasoning for your conclusion.
Problem 4 Essential Bandwidth of a Signal (20 points)

For almost all communication signals most of the signal energy is contained in a bandwidth $B$ Hz. The bandwidth $B$ is sometimes called the essential bandwidth of the signal. The criterion for selecting the value of $B$ depends upon the application. A common criterion is to select 95% of the signal energy to be contained in the communication system’s bandwidth. For this problem we make use of the results you obtained in working Problem 2 above.

Given the time-domain signal of a decaying exponential, restricted to positive times only, estimate the essential bandwidth $B$ (in radians/second) required to contain 95% of the signal’s energy by using the results obtained in Problem 2.

To work this problem, you must know the following: The energy spectral density (ESD) of the input signal $f(t)$ is given by

$$\Psi_F = |F(f)|^2.$$  

To obtain the total energy over a frequency band we must integrate over the band.

Starting with the result you obtained for $|F(\omega)|$ in Problem 2, determine the essential bandwidth $B$ (in radians/second) for the signal $f(t)$ in Problem 2 with the requirement that it contains 95% of the signal energy within this bandwidth.
**Problem 5 RLC Resonator** (20 points) [This should be review for you.]

One possible frequency selective circuit is a simple *LRC* resonator as schematically drawn below. The resonant frequency of the circuit is

\[ f_{\text{resonance}} = \frac{1}{\sqrt{LC}} \]

The *Quality factor* of a resonant circuit (*Q*-factor for short) is defined as the resonant frequency divided by the half-power (or -3 dB) bandwidth; in symbols it is

\[ Q = \frac{f_{\text{resonance}}}{B} \]

An *RLC* circuit finds use for selecting communication bands by tuning to the desired carrier frequency. It is also used for “demodulation” of amplitude modulation (AM) communication signals, “modulation” of frequency modulation (FM) communication signals and for establishing oscillation frequencies in local oscillators as used in superheterodyne receivers. Consider the parallel *RLC* circuit as shown below:

(a) Derive the transfer function \( H(\omega) \) for this parallel RLC circuit. Assume the sinusoidal steady-state in deriving the transfer function. We define \( H(\omega) \) as the ratio of the current \( i_R \) flowing through the resistor divided by the input current \( i(t) \).
(b) Derive the half-power bandwidth $B$ (i.e., the -3-dB bandwidth forming the frequency band between the -3 dB frequencies) for this circuit. Express bandwidth $B$ as a function of $R$, $L$ and $C$.

(c) Starting from the definition of $Q$-factor at the beginning of the problem statement; derive an expression for $Q$ as a function of the circuit parameters.