Problem 1  FM Signal Driving Square-Law Device  (15 points)

Suppose we apply an FM signal as the input $v_{in}$ to a square-law (non-linear) device with output as given by

$$v_{out} = a_{sq} \left(v_{in}\right)^2$$

where $a_{sq}$ is a constant.

If the FM signal passes through the square-law device, how does the frequency deviation $\Delta f$ change?

Solution:

Let $\varphi_{FM}(t) = v_{in} = A_c \cos \left( \omega_c t + k_f \int_0^t m(\lambda) d\lambda \right)$

$$v_{out} = a_{sq} \left(v_{in}\right)^2 = a_{sq} A_c^2 \cos^2 \left( \omega_c t + k_f \int_0^t m(\lambda) d\lambda \right)$$

$$v_{out} = \frac{a_{sq} A_c^2}{2} \left( 1 + \cos \left( 2\omega_c t + 2k_f \int_0^t m(\lambda) d\lambda \right) \right)$$

Therefore, the frequency deviation $\Delta f$ is doubled.

Problem 2  Zero Crossings in PM and FM  (30 points)

Consider a modulating wave $m(t)$ that increases linearly with time $t$, starting at $t = 0$. Mathematically this is expressed as

$$m(t) = \begin{cases} 
    at, & \text{for } t \geq 0 \\
    0, & \text{for } t < 0 
\end{cases}$$
where constant $a$ is the slope parameter of the ramp (see figure below). In this problem we study the zero-crossings of the both PM and FM waves produced by message signal $m(t)$ given the following parameters:

$$f_c = \frac{1}{4} \text{ Hz}$$
$$a = 1 \text{ volt/sec}$$

(a) **Phase Modulation** ($k_p = \pi/2$ radians/volt and $A_C = 1$ volt)

$$\varphi_{PM}(t) = \begin{cases} A_C \cos(2\pi f_c + k_p at), & \text{for } t \geq 0 \\ A_C \cos(2\pi f_c), & \text{for } t < 0 \end{cases}$$

When the angle in the argument of the cosine is an odd multiple of $\pi/2$ the phase modulated wave $\varphi_{PM}(t)$ experiences a zero crossing. Let $t_n$ represent the instant in time ($n$ is an integer) for which $\varphi_{PM}(t)$ exhibits a zero crossing event. Show that (i.e., derive) the expression for $t_n$ is given by

$$t_n = \frac{\frac{1}{2} + n}{2f_c + \frac{k_p}{\pi}a}$$

and that this expression reduces to $t_n = \frac{1}{2} + n$ applying the above parametric values for $f_c$, $k_p$ and $a$.

**Solution:**

Zero crossings occur for the condition that the argument of the cosine equal $(\pi/2 + n\pi)$, for $n = 0, 1, 2, 3, \ldots$. We write the equation,

$$2\pi f_c t_n + k_p a t_n = \frac{\pi}{2} + n\pi; \quad n = 0, 1, 2, \ldots$$

Solving for $t_n$ gives
\[ t_n = \frac{\frac{1}{2} + n}{2f_C + \frac{k_p}{\pi} a} \] 

and plugging in parameter values reduces the expression to

\[ t_n = \frac{\frac{1}{2} + n}{2f_C + \frac{k_p}{\pi} a} = \frac{\frac{1}{2} + n}{2 \left( \frac{1}{4} \right) + \left( \frac{\pi}{2} \right) (1)} = \frac{\frac{1}{2} + n}{\frac{1}{2} + \frac{1}{2}} = \frac{\frac{1}{2} + n}{1} = \frac{1}{2} + n \]

for \( n = 0, 1, 2, 3, \ldots \).

(b) Frequency Modulation \((k_f = 1 \text{ hertz/volt and } A_C = 1 \text{ volt})\)

\[ \varphi_{FM}(t) = \begin{cases} A_c \cos \left( 2\pi f_c + \pi k_p a t^2 \right), & \text{for } t \geq 0 \\ A_c \cos \left( 2\pi f_c \right), & \text{for } t < 0 \end{cases} \]

Derive an expression for \( t_n \) for the FM case.

**Solution:**

Zero crossings occur for the condition that the argument of the cosine equal \((\pi/2 + n \pi)\), for \( n = 0, 1, 2, 3, \ldots \). We can write

\[ 2\pi f_c t_n + \pi k_f a \left( t_n \right)^2 = \frac{\pi}{2} + n\pi; \quad n = 0, 1, 2, \ldots \]

\[ -\left( \frac{1}{2} + n \right) \frac{\pi}{\pi k_f a} + 2\pi f_c \frac{t_n}{\pi k_f a} + t_n^2 = 0 \]

Using the quadratic formula and taking only the positive square root term,
\[
\begin{align*}
t_n &= \frac{-2 f_c}{k_f a} + \sqrt{\left(\frac{2 f_c}{k_f a}\right)^2 - 4 \left(-\frac{1}{2} + n\right) \left(\frac{1}{2} + n\right) \left(\frac{1}{2} + n\right)} \\
t_n &= \frac{1}{k_f a} \left[ -f_c + \sqrt{f_c^2 + k_f a \left(\frac{1}{2} + n\right)} \right]; \quad n = 0, 1, 2, \ldots \\
\text{Substituting,} \\
t_n &= \frac{1}{4} \left[ -1 + \sqrt{9 + 8n} \right]; \quad n = 0, 1, 2, \ldots
\end{align*}
\]

(c) On the graph below sketch the general behavior of both \( \varphi_{PM}(t) \) and \( \varphi_{FM}(t) \) for \( t < 0 \) and \( t \geq 0 \). Draw conclusions.
Solution:

Your plot should look something like this.

General comments:

(1) For phase modulation, the ramp adds phase to the argument of the cosine so the frequency for \( t > 0 \) appears to be a higher frequency but is constant because of the linear increase in phase.

(2) For frequency modulation, the linear increase in \( m(t) \) appears to be a continuously increasing frequency for \( t > 0 \).
Problem 3  FM Modulator  (25 points)

The output of an FM modulator is given by
\[ \varphi_{FM}(t) = 20 \cdot \cos \left( \omega_c t + k_f \int_0^t m(\lambda) d\lambda \right) \]

where \( k_f = 20\pi \) radians/volt and \( m(t) \) is as shown below.

(a) Find the phase deviation \( \phi(t) \) as a function of time and sketch its shape.

Solution:

The output is of the general form,
\[ \varphi_{FM}(t) = A_c \cos \left( \omega_c t + \phi(t) \right), \]

where \( \phi(t) \) is the phase deviation. Thus,
\[ \phi(t) = 20\pi \int_0^t m(\lambda) d\lambda = \begin{cases} 0, & t < 0 \\ 100\pi t, & 0 \leq t \leq 2 \\ 200\pi, & t > 2 \end{cases} \]

The sketch for \( \phi(t) \) is as shown below:
(b) Find the frequency deviation $\Delta f$ as a function of time and sketch its shape.

**Solution:**

To determine $\Delta f$ in hertz, we differentiate the phase deviation $\phi(t)$ and divide by $2\pi$, henceforth

$$\Delta f = \frac{\frac{d\phi(t)}{dt}}{2\pi} = k_j m(t)$$

$$\Delta f = \frac{100\pi}{2\pi} = 50 \text{ Hz} \quad \text{for} \quad 0 \leq t \leq 2 \text{ sec} \quad \Leftarrow$$

and it is 0 otherwise

Sketch:

![Sketch of frequency deviation](image)

(c) Find the peak frequency deviation value in hertz.

**Solution:**

From the figure in part (b) immediately above it is obvious that the peak frequency deviation value is 50 Hz. \Leftarrow

(d) Find the peak phase deviation value in radians.

**Solution:**

From the figure in part (a) the peak phase deviation occurs at $t = 2$ seconds and is $200\pi$ radians. \Leftarrow
(e) What is the modulator’s output power? Assume into a one ohm resistance.

Solution:

\[
P = \frac{(20 \text{ V})^2}{2(1 \Omega)} = 200 \text{ watts}
\]

Problem 4 Sensitivity of FM Slope Detector (30 points)

One of the FM discriminators (detectors) we covered in our lectures on FM demodulation is the FM slope detector (slide 82 in the slide set on Angle Modulation). The circuit schematic below shows a parallel \( RLC \) resonant circuit we use in this problem to investigate the sensitivity of a slope detector.

Assume the incoming FM signal is modeled as a current source (say the current from an antenna). Of course, the sensitivity of this slope detector (used as a frequency to amplitude converter) depends upon the operating frequency – that is the offset between the carrier frequency \( f_C \) of the FM signal and the resonant frequency \( f_0 \) of the \( RLC \) circuit.

The input current flows into the input impedance \( Z(j\omega) \) of the resonant circuit which is given by the expression,
\[
|Z(j\omega)| = \frac{R}{\sqrt{1 + Q^2 \left[ \frac{f}{f_0} - \frac{f_0}{f} \right]^2}}
\]

where \( Q = 2\pi f_0 RC = \frac{R}{2\pi f_0 L} \) and \( (2\pi f_0)^2 = \omega_0^2 = \frac{1}{LC} \)

In this problem we want to estimate the sensitivity of the slope detector in converting the frequency variation (in Hz) into an amplitude variation (in volts). Given an FM slope detector with \( Q = 10 \), resistance \( R = 500 \) ohms, and resonant frequency \( f_0 \), estimate the detector’s sensitivity. To do this we choose to operate the slope detector at the carrier frequency \( f_C \) that is at 90% of the resonant frequency.

A way is to estimate the sensitivity is to evaluate the slope detector at two closely spaced frequencies, say at \( f_C + \Delta f \) and \( f_C - \Delta f \), with the frequency deviation \( \Delta f \) being small. To do this, choose \( f_C + \Delta f = 0.91 f_0 \) and \( f_C - \Delta f = 0.89 f_0 \). Thus, \( 2\Delta f \) is the change in frequency to be used in our calculation.

(a) With \( Q = 10 \) and \( R = 500 \) ohms, find the max and min values of the output voltage given a peak drive current \( I_{FM} \) equal to 10 mA. This will allow you to determine the change in voltage for a change in frequency. We assume here that \( f_C \) is nominally 90% of \( f_0 \). The sensitivity \( S \) is then expressed as the difference in output voltage \( (= I_{FM} \times |Z(j\omega)|) \). Express your answer for sensitivity \( S \) in volts per MHz. Assume that \( f_C \) is 100 MHz.
Solution:

We know that $Q = 10$, $R = 500$ ohms and $f_C = 100$ MHz. Using the equation for $|Z(j\omega)|$ allows us to calculate the change in impedance for the two frequency points ($f_C + \Delta f = 0.91f_0$ and $f_C - \Delta f = 0.89f_0$).

| $f$    | $\left[\frac{f}{f_0} - \frac{f_0}{f}\right]$ | $\left[\frac{f}{f_0} - \frac{f_0}{f}\right]^2$ | $|Z(j\omega)| = \frac{R}{\sqrt{1 + Q^2\left[\frac{f}{f_0} - \frac{f_0}{f}\right]^2}}$ | Voltage     |
|--------|--------------------------------|----------------|--------------------------------------------------------------------------------|-------------|
| $f = 0.89f_0$ | -0.2356 | 0.05457 | 196.8 $\Omega$ | 1.968 V                  |
| $f = 0.91f_0$ | -0.1889 | 0.03568 | 233.9 $\Omega$ | 2.339 V                  |

The voltage variation over approximately 2% of $f_0$ is $(2.339 V - 1.968 V = 0.371 V)$. We can determine the frequency range from

$$\frac{f_C + \Delta f}{0.91} = f_0 = \frac{f_C - \Delta f}{0.89} \quad \text{or} \quad (-0.02247)f_C = -(2.02247)\Delta f$$

which gives $\Delta f = 0.011111 \times f_C = 1.111$ MHz
The range for the frequency is $2\Delta f = 2.222$ MHz and therefore the sensitivity $S$ is

$$S = \frac{(0.371 \text{ Volt})}{(2.222 \text{ MHz})} = 0.167 \text{ Volts/MHz}$$

(a) What is the resonant frequency $f_0$ of the slope detector?

**Answer:**

Using

$$\frac{f_c + \Delta f}{0.91} = f_0; \quad \text{With} \ \Delta f = 1.111 \text{ MHz and} \ f_c = 100 \text{ MHz},$$

$$f_0 \approx 111.1 \text{ MHz}$$

We are approximately 11% below the resonance frequency in using the RLC circuit as a slope detector.

(b) What is the bandwidth $BW$ of the resonant circuit with $Q = 10$?

**Answer:**

We can calculate the bandwidth $BW$ from the simple expression of

$$BW = \omega_U - \omega_L = \frac{\omega_0}{Q} = \frac{111.0 \text{ MHz}}{10} = 11.11 \text{ MHz}$$