Angle Modulation (Phase & Frequency Modulation)
EE442 Lecture 7

Spring Semester

Angle Modulation

Two forms of angle modulation

Frequency modulation (FM)
Phase modulation (PM)

Issued 1983

Start reading Chapter 4 in Agbo & Sadiku
**Applications for Various Modulation Techniques**

<table>
<thead>
<tr>
<th>Application</th>
<th>Type of Modulation</th>
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<tbody>
<tr>
<td>AM broadcast radio</td>
<td>AM</td>
</tr>
<tr>
<td>FM broadcast radio</td>
<td>FM</td>
</tr>
<tr>
<td>FM stereo multiplex sound</td>
<td>DSB (AM) and FM</td>
</tr>
<tr>
<td>TV sound</td>
<td>FM</td>
</tr>
<tr>
<td>TV picture (video)</td>
<td>AM, VSB</td>
</tr>
<tr>
<td>TV color signals</td>
<td>Quadrature DSB (AM)</td>
</tr>
<tr>
<td>Cellular telephone</td>
<td>FM, FSK, PSK</td>
</tr>
<tr>
<td>Cordless telephone</td>
<td>FM, PSK</td>
</tr>
<tr>
<td>Fax machine</td>
<td>FM, QAM (AM plus PSK)</td>
</tr>
<tr>
<td>Aircraft radio</td>
<td>AM</td>
</tr>
<tr>
<td>Marine radio</td>
<td>FM and SSB (AM)</td>
</tr>
<tr>
<td>Mobile and handheld radio</td>
<td>FM</td>
</tr>
<tr>
<td>Citizens band radio</td>
<td>AM and SSB (AM)</td>
</tr>
<tr>
<td>Amateur radio</td>
<td>FM and SSB (AM)</td>
</tr>
<tr>
<td>Computer modems</td>
<td>FSK, PSK, QAM (AM plus PSK)</td>
</tr>
<tr>
<td>Garage door opener</td>
<td>OOK</td>
</tr>
<tr>
<td>TV remote control</td>
<td>OOK</td>
</tr>
<tr>
<td>VCR</td>
<td>FM</td>
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<tr>
<td>Family Radio service</td>
<td>FM</td>
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</table>

Not a complete list of applications.

We have studied AM, next is FM and PM.
With few exceptions, Phase Modulation (PM) is used predominantly in digital communication.

The rate of change in the phase is equal to the frequency.
Illustrating AM, PM and FM Signals

- **Carrier Wave**
  - $m(t)$
  - Carrier signal

- **Modulating Signal $m(t)$**

- **AM Modulated Signal**

- **PM Modulated Signal**
  - $\sim \frac{dm(t)}{dt}$

- **FM Modulated Signal**
  - $\sim m(t)$

Reference:
Lathi & Ding
Focus Upon an FM Signal Modulated by a Single-Tone

Single-tone modulating signal

$\mathcal{M}(t)$
Comparing AM, PM and FM for a Ramp $m(t)$

carrier $\cos(\omega_c t)$

message $m(t)$

amplitude modulation

phase modulation

frequency modulation

Some Observations on FM and PM Waveforms

1. Both FM and PM waveforms are identical except for a time shift, given that \( m(t) \) is a sinusoidal signal.

2. For FM, the maximum frequency deviation occurs when modulating signal is at its peak values (i.e., at \( +m_p \) and \( -m_p \)).

3. For PM, the maximum frequency deviation takes place at the zero crossings of the modulating signal \( m(t) \).

4. It is usually difficult to know from looking at a waveform whether the modulation is FM or PM.

5. The message resides in the zero-crossings alone, provided the carrier frequency is large.

6. The modulated waveform doesn’t resemble the message waveform.

Reference: Carlson & Crilly, 5th ed., Section 5.1, pages 208 to 212.
Advantages of Angle Modulation

1. Angle modulation is **resistant to propagation-induced selective fading** because amplitude variations are not important.

2. Angle modulation is very efficient in **rejecting interference** (i.e., it minimizes the effect of amplitude noise on the signal transmission).

3. Angle modulation allows for more **efficient use of transmitter power**.

4. Angle modulation is capable of **handling a greater dynamic range** in the modulating signal without distortion as would occur in AM.

5. Wideband FM gives significant **improvement in the signal-to-noise ratio** at the output and is proportional to the square of the modulation index $\beta$.

$$\beta = \frac{\Delta f}{B} \begin{cases} B & \text{Bandwidth} \\ \Delta f & \text{Frequency deviation} \end{cases}$$
Phase-Frequency Relationship When Frequency is Constant

\[ \phi(t) = A_C \cos(\theta(t)) \]

\( \theta(t) \) is generalized angle

\[ \phi(t) = A_C \cos(\omega_C t + \theta_0) \]

\( \omega_C t + \theta_0 \)

\( \theta_0 \) is constant

Slope: \[ \omega_i(t) = \frac{d\theta(t)}{dt} \bigg|_{t=t_i} = \omega_C \]
Concept of Instantaneous Frequency

\[ \varphi(t) = A_C \cos(\theta(t)) \]

\( \theta(t) \) is generalized angle

\[ \varphi(t) = A_C \cos(\omega_C t + \theta_0) \]

\( \theta_0 \) is constant

Slope: \( \omega_i(t) = \left. \frac{d\theta(t)}{dt} \right|_{t=t_i} > \omega_C \)
Angle Modulation Gives PM and FM

\[
\omega_i(t) = \frac{d\theta(t)}{dt} \bigg|_{t=t_i} \quad \text{and} \quad \theta(t) = \int_{-\infty}^{t} \omega_i(\lambda) \, d\lambda
\]

Frequency modulation and phase modulation are closely related!
# Comparing Frequency Modulation to Phase Modulation

<table>
<thead>
<tr>
<th>No.</th>
<th>Frequency Modulation (FM)</th>
<th>Phase Modulation (PM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Frequency deviation is proportional to modulating signal $m(t)$</td>
<td>Phase deviation is proportional to modulating signal $m(t)$</td>
</tr>
<tr>
<td>2</td>
<td>Noise immunity is superior to PM (and of course AM)</td>
<td>Noise immunity better than AM, but not FM</td>
</tr>
<tr>
<td>3</td>
<td>Signal-to-noise ratio (SNR) is better than PM (and of course AM)</td>
<td>Signal-to-noise ratio (SNR) is not quite as good as with FM</td>
</tr>
<tr>
<td>4</td>
<td>FM is widely used for commercial broadcast radio (88 MHz to 108 MHz)</td>
<td>PM is primarily used for mobile radio services</td>
</tr>
<tr>
<td>5</td>
<td>Modulation index is proportional to modulating signal $m(t)$ as well as the modulating frequency $f_m$</td>
<td>Modulation index is proportional to modulating signal $m(t)$</td>
</tr>
</tbody>
</table>
FM has superior noise immunity compared to AM

FM has better noise (or RFI) rejection than AM, as shown in this dramatic New York publicity demonstration by General Electric in 1940. The radio contained both AM and FM receivers. With a million-volt arc as a source of interference behind it, the AM receiver produced only a roar of static, while the FM receiver clearly reproduced a music program from Armstrong's experimental FM transmitter W2XMN in New Jersey.

https://en.wikipedia.org/wiki/Frequency_modulation

Note: RFI stands for radio frequency interference.
Phase Modulation (PM)

\[\theta_i(t) = \omega_C t + \theta_0 + k_p m(t) ; \quad \text{Usually we set } \theta_0 = 0,\]

\[\varphi_{PM}(t) = A_C \cos(\omega_C t + k_p m(t))\]

The instantaneous angular frequency (in radians/second) is

\[\omega_i(t) = \frac{d\theta_i(t)}{dt} = \omega_C + k_p \frac{dm(t)}{dt} = \omega_C + k_p m'(t)\]

In phase modulation (PM) the instantaneous angular frequency \(\omega_i\) varies linearly with the time derivative of the message signal \(m(t)\) [denoted here by \(m'(t)\)].

\(k_p\) is the phase-deviation (sensitivity) constant. Units: radians/volt [Actually it is radians/unit of the parameter \(m(t)\).]
Frequency Modulation (FM)

But in frequency modulation the instantaneous angular frequency \( \omega_i \) varies linearly with the modulating signal \( m(t) \),

\[
\omega_i(t) = \omega_C + k_f m(t)
\]

\[
\theta_i(t) = \int_{-\infty}^{t} \left[ \omega_C + k_f m(\lambda) \right] d\lambda = \omega_C t + k_f \int_{-\infty}^{t} m(\lambda) d\lambda
\]

\( k_f \) is frequency-deviation (sensitivity) constant. Units: radians/volt-sec.

Then

\[
\phi_{FM}(t) = A_C \cos \left( \omega_C t + k_f \int_{-\infty}^{t} m(\lambda) d\lambda \right)
\]

FM and PM are related to each other. In PM the angle is directly proportional to \( m(t) \). In FM the angle is directly proportional to the integral \( \int m(t) dt \).
Summary

Message signal is $m(t)$

Definition: Instantaneous frequency is $\omega_i(t) = \frac{d\theta_i(t)}{dt}$

<table>
<thead>
<tr>
<th>Angle</th>
<th>Phase Modulation</th>
<th>Frequency Modulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_i(t) = \omega_C t + k_p m(t)$</td>
<td>$\theta_i(t) = \omega_C t + k_f \int_{-\infty}^{t} m(\lambda) d\lambda$</td>
<td></td>
</tr>
<tr>
<td>$\omega_i = \omega_C + k_p \frac{dm(t)}{dt}$</td>
<td>$\omega_i = \omega_C + k_f m(t)$</td>
<td></td>
</tr>
</tbody>
</table>

In phase modulation $m(t)$ drives the time variation of phase $\theta_i$.
In frequency modulation $m(t)$ drives the time variation of frequency $f_C$. 
A Pictorial View of FM and PM Generation

We require that $H(j\omega)$ be a reversible (or invertible) operation so that $m(t)$ is recoverable.

Webo & Sadiku
Figure 4.1
p. 160
Both FM and PM Generation are Nonlinear Processes

Consider a phase modulated signal:

Let \( s(t) = A_C \cos(\omega_C t + k_p [m_1(t) + m_2(t)]) \)

If \( s_1(t) = A_C \cos(\omega_C t + k_p m_1(t)) \), and

\( s_2(t) = A_C \cos(\omega_C t + k_p m_2(t)) \)

It then holds that

\( s_1(t) + s_2(t) \neq s(t) \leq additivity \text{ fails} \)

So PM can't be linear.

The same argument holds for FM.
Modulation Index $\beta$ for Angle Modulation

Let the peak values of the message signal $m(t)$ and its first derivative $m'(t)$ be represented by

\[
\text{Peak value of } m(t) = m_p = \frac{1}{2}(m_{\max} - m_{\min})
\]

\[
\text{Peak value of } m'(t) = \frac{dm(t)}{dt} = m'_p
\]

**Frequency Deviation** is the maximum deviation of the instantaneous modulated carrier frequency relative to the unmodulated carrier frequency. It is symbolically represented by either $\Delta \omega$ or $\Delta f$.

FM: \[ \Delta \omega = k_f m_p \quad \text{or} \quad \Delta f = \frac{k_f m_p}{2\pi} \]

PM: \[ \Delta \omega = k_p m'_p \quad \text{or} \quad \Delta f = \frac{k_p m'_p}{2\pi} \]

The ratio of the frequency deviation $\Delta f$ to the message signal’s bandwidth $B$ is called the Frequency Deviation Ratio or the **Modulation Index**, and is denoted by $\beta$ (unitless).

\[ \beta = \frac{\Delta f}{B} = \frac{\Delta \omega}{2\pi B} \]
Equations for FM Wave with Single-Tone Modulation

Carrier signal
\[ A_c \cos(\omega_c t) \] (volts)

Carrier frequency
\[ \omega_c = 2\pi f_c \] (radians/sec)

Modulating wave \( m(t) \)
\[ A_m \cos(\omega_m t) \] Single-tone modulation

Modulating frequency
\[ \omega_m = 2\pi f_m \] (radians/sec)

Deviation sensitivity
\[ k_f \] (radians/volt-second)

Frequency deviation
\[ \Delta \omega = k_f A_m \] (radians/sec)

Modulation Index
\[ \beta = \frac{\Delta f}{f_m} = \frac{\Delta \omega}{\omega_m} = \frac{k_f A_m}{\omega_m} \] (unitless)

Instantaneous frequency
\[ f_i = f_c + k_f \frac{A_m}{2\pi} \cos(\omega_m t) = f_c + \Delta f \cos(\omega_m t) \]

Remember
\[ \varphi_{FM}(t) = A_c \left[ \cos \left( \omega_c t + k_f \left( \int\limits_{-\infty}^{t} m(\lambda) d\lambda \right) \right) \right] \], generally

Tone modulated wave
\[ \varphi_{FM}(t) = A_c \left[ \cos \left( \omega_c t + k_f A_m \frac{\sin(\omega_m t)}{\omega_m} \right) \right] \]

or
\[ \varphi_{FM}(t) = A_c \left[ \cos(\omega_c t + \beta \sin(\omega_m t)) \right] \]
## Summary of Mathematical Equations for FM and PM

<table>
<thead>
<tr>
<th>Type of Modulation</th>
<th>Modulating Signal</th>
<th>Angle Modulated Wave</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase modulation</td>
<td>$m(t)$</td>
<td>$A_C \cdot \cos\left(\omega_C t + k_p m(t)\right)$</td>
</tr>
<tr>
<td>Frequency modulation</td>
<td>$m(t)$</td>
<td>$A_C \cdot \cos\left(\omega_C t + k_f \int_{-\infty}^{t} m(\lambda) d\lambda \right)$</td>
</tr>
<tr>
<td>Phase modulation</td>
<td><strong>Tone:</strong> $m(t) = A_m \cdot \cos(\omega_m t)$</td>
<td>$A_C \cdot \cos\left(\omega_C t + k_p A_m \cos(\omega_C t)\right)$</td>
</tr>
<tr>
<td>Frequency modulation</td>
<td><strong>Tone:</strong> $m(t) = A_m \cdot \cos(\omega_m t)$</td>
<td>$A_C \cdot \cos\left(\omega_C t + \frac{k_f A_m}{\omega_m} \sin(\omega_C t)\right)$</td>
</tr>
</tbody>
</table>

$$\beta \triangleq \frac{k_f A_m}{\omega_m}$$
Example

- A single-tone FM signal is

\[ \varphi_{FM}(t) = 10 \left[ \cos \left( 2\pi (10^6) t + 8 \sin(2\pi (10^3) t) \right) \right] \]

Determine

a) the carrier frequency \( f_C \)

b) the modulation index \( \beta \)

c) the peak frequency deviation \( \Delta f \)
Solution to Example

Start with the basic FM equation:
\[ \varphi_{FM}(t) = A_C \left[ \cos(2\pi f_C t + \beta \sin(2\pi f_m t)) \right] \]

Compare this to
\[ \varphi_{FM}(t) = 10 \left[ \cos\left(2\pi (10^6) t + 8 \sin(2\pi (10^3)t)\right)\right] \]

(a) We see that \( f_C = 1,000,000 \) Hz & \( f_m = 1000 \) Hz.
(b) The modulation index is \( \beta = 8 \).
(c) The peak deviation frequency \( \Delta f \) is

\[ \Delta f = \beta \cdot f_m = 8 \cdot 1000 = 8,000 \text{ Hz} \]

Note: \( \Delta f / f_C \) is 0.008 or 0.8 % deviation frequency to carrier frequency.
Average Power of a FM or PM Wave

The amplitude $A_C$ is constant in a phase modulated or a frequency modulated signal. RF power does not depend upon the frequency or the phase of the waveform.

$$\phi_{FM \text{ or } PM}(t) = A_C \cos[\omega_c t + g(k_k, m(t))]$$

Average Power $= \frac{A_C^2}{2}$ (always)

This is a result of FM and PM signals being constant amplitude.

Note: $k_k$ becomes $k_f$ for FM and $k_p$ for PM.
Average Power of a FM or PM Wave

Problem:

Consider an angle modulated signal given by

\[ \phi(t) = 6 \cdot \left[ \cos \left( 2\pi \times 10^6 t + 2 \cdot \sin(8000\pi t) \right) \right] \text{ volts} \]

What is the average power of this signal?

Solution:

Average power \( P_C = \frac{A_C^2}{2} \) where \( A_C = 6 \) volts

Therefore, \( P_C = \frac{6^2}{2} = \frac{36}{2} = 18 \) watts (assumes 1 ohm resistance)

Note that the result does not depend upon it being FM or PM.
## Comparison of FM (or PM) to AM

<table>
<thead>
<tr>
<th>#</th>
<th>Frequency Modulation (FM)</th>
<th>Amplitude Modulation (AM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>FM receivers have better noise immunity</td>
<td>AM receivers are very susceptible to noise</td>
</tr>
<tr>
<td>2</td>
<td>Noise immunity can be improved by increasing the frequency deviation</td>
<td>The only option in AM is to increase the transmission power</td>
</tr>
<tr>
<td>3</td>
<td>Bandwidth requirement is greater and depends upon modulation index</td>
<td>AM bandwidth is less than FM or PM and doesn’t depend upon a modulation index</td>
</tr>
<tr>
<td>4</td>
<td>FM (or PM) transmitters and receivers are more complex than for AM</td>
<td>AM transmitters and receivers are less complex than for FM (or PM)</td>
</tr>
<tr>
<td>5</td>
<td>All transmitted power is useful so FM is very efficient</td>
<td>Power is wasted in transmitting the carrier and double sidebands in DSB (but DSB-SC &amp; SSB addresses this)</td>
</tr>
</tbody>
</table>
AM, FM and PM Waveforms for Single-Tone $m(t)$

Review:

- Carrier signal
- Modulating Signal $m(t)$

Focus upon frequency

Angle Modulation

$AM$ $\sim dm(t)$

$PM$ $\sim \frac{dm(t)}{dt}$

$FM$ $\sim m(t)$

Reference: Lathi & Ding

Refer to slide 16
FM and PM Examples

Sketch FM and PM waveforms for the modulating signal $m(t)$. The constants $k_f$ and $k_p$ are $2\pi \times 10^5$ and $10\pi$, respectively. Carrier frequency $f_c = 100$ MHz.

$$f_i = f_c + \frac{k_f}{2\pi} m(t) = 1 \times 10^8 + 1 \times 10^5 \cdot m(t);$$

$$(f_i)_\text{min} = -1 \text{ and } (f_i)_\text{max} = 1$$

$$m'_{\text{min}} = -20,000 \text{ and } m'_{\text{max}} = 20,000$$

$$m'_{\text{min}} = 10^8 + 5(-20,000) = 99.9 \text{ MHz},$$

$$m'_{\text{max}} = 10^8 + 5(+20,000) = 100.1 \text{ MHz}$$

Fig. 5.4; p. 256 of 4th ed., Lathi & Ding
Digital Frequency Shift Keying is Related to FM

Sketch the FM waveform for the modulating signal \( m(t) \). The constant \( k_f \) is \( 2\pi \times 10^5 \). Carrier frequency \( f_c = 100 \) MHz.

Since \( m(t) \) switches from +1 to -1 and vice versa, the FM wave frequency switches between 99.9 MHz and 100.1 MHz. This is called **Frequency Shift Keying (FSK)** and is a digital communication format.

\[
f_i = f_c + \frac{k_f}{2\pi} m(t) = 1\times10^8 + 1\times10^5 m(t)
\]

*Fig. 5.5; p. 258 of 4th ed., Lathi & Ding*
Example – continued

Sketch the PM waveform for the modulating signal \( m(t) \) from prior slide. The constant \( k_p \) equals \( \pi/2 \). Carrier frequency \( f_c = 100 \text{ MHz} \).

Evaluate the instantaneous jumps by considering:

\[
\varphi_{PM}(t) = A_C \cos \left[ \omega_c t + k_p m_d(t) \right] = A_C \cos \left[ \omega_c t + \frac{\pi}{2} m_d(t) \right]
\]

\( k_p m_d(t) \sim (-\pi, \pi) \)

- \( \varphi_{PM}(t) = A_C \sin(\omega_c t) \) when \( m(t) = -1 \)
- \( \varphi_{PM}(t) = -A_C \sin(\omega_c t) \) when \( m(t) = 1 \)

where jump in \( m_d(t) = (1) - (-1) = 2 \) or \( (-1) - (1) = -2 \)

This is carrier PM by a digital signal – it is **Phase Shift Keying (PSK)** because the digital data is represented by phase of the carrier wave.

**Fig. 5.5; p. 258 of 4th ed., Lathi & Ding**
Generalized Angle Modulation

Agbo & Sadiku; Section 4.2 & 4.3 on pages 158 to 168

Start with equation (4.8) on page 159, which is

\[ \phi_A(t) = A_C \cdot \cos[\omega_C t + k \cdot \gamma(t)] \quad \text{where} \quad \gamma(t) = m(t) \ast h(t) = \int_{-\infty}^{t} m(\lambda)h(t-\lambda)d\lambda \]

with \( h(t) = \delta(t) \) for PM; \( h(t) = u(t) \) for FM

Suppose we use the exponential carrier \( A_C e^{j\omega_C t} \) instead of \( A_C \cos(\omega_C t) \), then the form for generalized angle modulation becomes

\[ \phi_A(t) = A_C \cdot e^{j(\omega_C t + k\gamma(t))} = A_C \cdot e^{j(\omega_C t)} \cdot e^{jk\gamma(t)} \]

where \( k \rightarrow k_p \) for PM; \( k \rightarrow k_f \) for FM

\[ \phi_{PM}(t) = \text{Re}[\phi_A(t)] = \text{Re}\left[ A_C \cdot e^{j(\omega_C t)} \cdot e^{jk_p\gamma(t)} \right] \quad \text{where} \quad \gamma(t) = m(t) \ast \delta(t) = m(t) \]

and

\[ \phi_{FM}(t) = \text{Re}[\phi_A(t)] = \text{Re}\left[ A_C \cdot e^{j(\omega_C t)} \cdot e^{jk_f\gamma(t)} \right] \quad \text{where} \quad \gamma(t) = \int_{-\infty}^{t} m(\lambda)d\lambda \]
Generalized Angle Modulation (continued)

Consider first Frequency Modulation (FM),

\[
\phi_{FM}(t) = \Re \left[ \phi_A(t) \right] = \Re \left[ A_C \cdot e^{j\omega_C t} \sum_{n=0}^{\infty} \frac{j^n k_f^n \gamma^n(t)}{n!} \right]
\]

\[
\phi_{FM}(t) = \Re \left[ A_C \cdot e^{j\omega_C t} \left(1 + jk_f \gamma(t) - \frac{k_f^2 \gamma^2(t)}{2!} + \frac{jk_f^3 \gamma^3(t)}{3!} - \cdots \right) \right]
\]

Now take the real part of the expression above,

\[
\phi_{FM}(t) = A_C \left[ \cos(\omega_C t) - k_f \gamma(t) \sin(\omega_C t) - \frac{k_f^2 \gamma^2(t)}{2!} \cos(\omega_C t) + \cdots \right]
\]

Note: \( m(t) \) has a bandwidth = \( B \) Hz and \( \gamma(t) \) has a bandwidth = \( B \) Hz, but \( \gamma^n(t) \) has a bandwidth = \( nB \) Hz; as \( n \to \infty \), bandwidth \( \to \infty \)

The instantaneous frequency deviations are symmetrical about carrier frequency \( \omega_C \), thus FM is double side-banded. The effective FM bandwidth = \( 2nB \) Hz.
Generalized Angle Modulation (continued)

Agbo & Sadiku; Section 4.2 & 4.3 on pages 158 to 168

Consider the case where \( k_f \) is small, meaning that \( |k_f \gamma(t)| \ll 1 \). It is commonly referred to as **narrowband FM** (NBFM). We take only the first two terms in the expansion for \( \phi_{FM}(t) \).

\[
\phi_{FM}(t) \approx A_C \left[ \cos(\omega_C t) - k_f \gamma(t) \sin(\omega_C t) \right]
\]

**FM:** \( \phi_{FM}(t) \approx A_C \cdot \cos(\omega_C t) - A_C k_f \int_{-\infty}^{t} m(\lambda)d\lambda \sin(\omega_C t) \)  Equation (4.15)

By analogy, we can apply same analysis for Phase Modulation (PM). For PM, if \( k_p \) is small, then \( |k_p \gamma(t)| \ll 1 \). This is known as **narrowband PM** (NBPM).

\[
\phi_{PM}(t) \approx A_C \cdot \cos(\omega_C t) - A_C k_p \left[ m(t) \right] \sin(\omega_C t) \]  Equation (4.16)

Using these results allows us to generate narrowband FM and PM with the block diagrams on the next slide (slide #34):
Generation of Narrowband FM and PM

\[ m(t) \int k_f \sum \pi/2 -A_c \cdot \sin(\omega_c t) A_c \cdot \cos(\omega_c t) \]

\[ m(t) k_p \sum \pi/2 -A_c \cdot \sin(\omega_c t) A_c \cdot \cos(\omega_c t) \]

Agbo & Sadiku; Figure 4.5 on page 168
Modulation Index $\beta$ Parameter in Angle Modulation

Parameter $\beta$ is the modulation index for angle modulation. $\beta$ is used to differentiate between narrowband angle modulation and wideband angle modulation.

Narrowband angle modulation requires $\beta \ll 1$  (Typically $< 0.3$)
Wideband angle modulation requires $\beta \gg 1$  (Typically $> 5.0$)

Equivalently,

Narrowband angle modulation requires $\Delta f \ll B$
Wideband angle modulation requires $\Delta f \gg B$

Comments:
1. Narrowband FM has about the same bandwidth as that of AM.
2. Commercial (broadcast) FM is wideband FM (required due to its superior noise performance).
3. Why even consider narrowband FM? Two reasons:
   a. NBFM is easier to generate than WBFM.
   b. It is used as in initial process step in generating WBFM.
Narrowband FM with Tone Modulation

Let \( m(t) = A_m \cos(\omega_m t) \), then \( m_p = A_m \); \( \omega_m = 2\pi B \); and \( \beta = \frac{k_f m_p}{\omega_m} = \frac{k_f A_m}{\omega_m} \)

Then \( k_f \int_{-\infty}^{t} m(\lambda)d\lambda \) = \( k_f \int_{-\infty}^{t} A_m \cos(\omega_m \lambda)d\lambda \) = \( \frac{k_f A_m}{\omega_m} \cdot \sin(\omega_m t) \)

The time-domain NBFM signal is
\[
\phi_{FM}(t) \cong A_C \cos(\omega_C t) - \beta A_C \sin(\omega_m t) \cdot \sin(\omega_C t)
\]
Equation (4.18)

\[
\phi_{FM}(t) \cong A_C \cos(\omega_C t) + \frac{1}{2} \beta A_C \cos\left((\omega_C + \omega_m) t\right) - \frac{1}{2} \beta A_C \cos\left((\omega_C - \omega_m) t\right)
\]
Equation (4.19)

In comparing to AM:

The 2\textsuperscript{nd} term is the upper sideband and the 3\textsuperscript{rd} term is the lower sideband.
Narrowband FM (NBFM)

\[ \phi_{FM}(t) \approx A_C \cos(\omega_C t) + \frac{1}{2} \beta A_C \cos((\omega_C + \omega_m) t) - \frac{1}{2} \beta A_C \cos((\omega_C - \omega_m) t) \]

Tone modulation \( \sim \cos(\omega_m t) \)

\[ \phi_{FM}(t) \]

\[ A_C \]

\( \beta = 0.2 \)

Sidebands are in quadrature.

NBPM requires \( \beta \ll 1 \) radian (generally less than 0.3 radian)
Narrowband FM (NBFM)

\[ \omega_c \text{ rotates faster than } \omega_m \]

Phasor lengths adjust to keep constant \( A_c \).
Review: Phasor Interpretation of AM DSB with Carrier

\[ \omega_C \text{ rotates faster than } \omega_m \]

\[ \omega_m = |\omega_{us}| = |\omega_{ls}| \]

Spectrum:

DSB AM
Narrowband FM Example (Example 4.4)

**Exercise:** The message signal input to a modulator is \( m(t) = 4 \cdot \cos(2\pi \times 10^4 t) \) and the carrier is \( 10 \cdot \cos(\pi \times 10^8 t) \). If frequency modulation is performed with \( k_f = 1000\pi \), verify that the modulated signal meets the criteria of being narrowband FM. Also, obtain an expression for its spectrum and sketch this spectrum.

**Solution:**

Agbo & Sadiku; page 170

First we calculate the modulation index \( \beta \)

\[
\beta = k_f \frac{A_m}{\omega_m} = 1000\pi \left( \frac{4}{2\pi \times 10^4} \right) = 0.2; \quad \beta < 0.3 \Rightarrow \text{NBFM}
\]

\[
A_C = 10 \quad \text{thus,} \quad \frac{1}{2} \beta A_C = \frac{1}{2} (0.2)(10) = 1
\]

Using the equation from slide #37:

\[
\phi_{FM}(t) \cong A_C \cos(\omega_C t) + \frac{1}{2} \beta A_C \cos((\omega_C + \omega_m)t) - \frac{1}{2} \beta A_C \cos((\omega_C - \omega_m)t)
\]

\[
\phi_{FM}(t) \cong 10 \cdot \cos(\omega_C t) + \cos((\omega_C + \omega_m)t) - \cos((\omega_C - \omega_m)t)
\]
The corresponding expression for the spectrum becomes
\[ \Phi_{FM}(\omega) = 10\pi \left[ \delta(\omega + \omega_c) + \delta(\omega - \omega_c) \right] + \pi \left[ \delta(\omega + \omega_c + \omega_m) + \delta(\omega - \omega_c - \omega_m) \right] - \pi \left[ \delta(\omega + \omega_c - \omega_m) + \delta(\omega - \omega_c + \omega_m) \right] \]

where \( \omega_c = 10^8 \pi \) radians/sec and \( \omega_m = 2\pi \times 10^4 \) radians/sec
Wideband FM (WBFM)

WBFM requires $\beta >> 1$ radian (much more complicated)

For wideband FM we have a nonlinear process, with single-tone modulation:

$$\phi_{\text{WB}}(t) = \text{Re} \left[ A_C \exp(j\omega_C t + j\beta \sin(\omega_m t)) \right]$$

We need to expand the exponential in a Fourier series in order to analyze $\phi_{\text{FM}}(t)$. The solution has an expansion in Bessel functions:

$$\phi_{\text{WB}}(t) = A_C \sum_{n=-\infty}^{\infty} J_n(\beta) \cdot \cos(2\pi(f_C + nf_m)t)$$

where the coefficients $J_n(\beta)$ are Bessel functions.

Modulation Index

$$\beta = \frac{\Delta f}{f_m} = \frac{\Delta f}{B}$$

We will not cover this section in EE442 but rather focus upon the physical interpretation of FM spectrum spread.
**Digression: Bessel Functions (of the 1\textsuperscript{st} kind)**

Bessel functions have many applications: cylindrical waveguides, vibrational modes on circular membrane, FM modulation synthesis, acoustic vibrations, etc.

\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y = 0 \]

http://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html

https://www.cefns.nau.edu/~schulz/Bessel/J02.html
WBFM (or WBPM) Requires More bandwidth Than AM

Carrier Signal (frequency $f_c$)

Message Signal (frequency $f_m$)

Amplitude Modulated Signal

Frequency Modulated (FM) Signal
Single-Tone FM Spectra as Function of Modulation Index $\beta$

- **NBFM**
  - $\beta = 0.2$
  - $\beta = 1.0$

- **WBFM**
  - $\beta = 5$
  - $\beta = 10$

**Table:**

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>Number of Sidebands¶</th>
<th>Bandwidth</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>2</td>
<td>$2f_m$</td>
</tr>
<tr>
<td>0.3</td>
<td>4</td>
<td>$4f_m$</td>
</tr>
<tr>
<td>0.5</td>
<td>4</td>
<td>$4f_m$</td>
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<tr>
<td>1.0</td>
<td>6</td>
<td>$6f_m$</td>
</tr>
<tr>
<td>2.0</td>
<td>8</td>
<td>$8f_m$</td>
</tr>
<tr>
<td>5.0</td>
<td>16</td>
<td>$16f_m$</td>
</tr>
<tr>
<td>10.0</td>
<td>28</td>
<td>$28f_m$</td>
</tr>
</tbody>
</table>

¶Both upper and lower sidebands about $f_c$.

**Formula:**

$$\beta = \frac{\Delta f}{f_m} = \frac{\Delta \omega}{\omega_m}$$

$B_T$ or $BW$
Spectra of FM Signals

Δf increasing & f_m is constant

Single-tone
Modulation Index

\[ \beta = \frac{\Delta f}{f_m} = \frac{\Delta \omega}{\omega_m} \]

Δf is constant & f_m is decreasing

From A. Bruce Carlson, Communication Systems, An Introduction to Signals and Noise in Electrical Communication, 2nd edition, 1975; Chapter 6, Figure 6.5, Page 229.
Selecting an FM Station

Broadcast FM Radio covers from 88 MHz to 108 MHz
100 stations – 200 kHz spacing between FM stations

<table>
<thead>
<tr>
<th>Service Type</th>
<th>Frequency Band</th>
<th>Channel Bandwidth</th>
<th>Maximum Deviation</th>
<th>Highest Audio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commercial FM Radio</td>
<td>88.0 to 108.0 MHz</td>
<td>200 kHz</td>
<td>±75 kHz</td>
<td>15 kHz</td>
</tr>
</tbody>
</table>

Note: 0 dBu = 0.775 volt into 600 ohms (which is equivalent to 1 mW power delivered into the 600 ohm resistor)
Measured Spectrum of an FM Radio Signal

FM Radio Search

- IQ rate: 10M
- Carrier frequency: 94.7M
- Gain: 15
- Number of samples: 200k
- Active antenna: RX1

Voice modulation

Amplitude

200 kHz

Detected Stations

- 93.7M
- 94.7M
- 95.5M
- 96.7M
- 98.1M
- 98.9M

STOP
# Specifications for Some Commercial FM Transmissions

<table>
<thead>
<tr>
<th>Service Type</th>
<th>Frequency Band</th>
<th>Channel Bandwidth</th>
<th>Maximum Deviation</th>
<th>Highest Audio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commercial <strong>FM Radio Broadcast</strong></td>
<td>88.0 to 108.0 MHz</td>
<td>200 kHz</td>
<td>±75 kHz</td>
<td>15 kHz</td>
</tr>
<tr>
<td>Television Sound (analog)</td>
<td>4.5 MHz above the picture carrier frequency</td>
<td>100 kHz</td>
<td>±25 kHz monaural &amp; ±50 kHz stereo</td>
<td>15 kHz</td>
</tr>
<tr>
<td>Digital TV has replaced</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Public safety – Police, Fire, Ambulance, Taxi, Forestry, Utilities, &amp; Transportation</td>
<td>50 MHz and 122 MHz to 174 MHz</td>
<td>20 kHz</td>
<td>±5 kHz</td>
<td>3 kHz</td>
</tr>
<tr>
<td>Amateur, CE class A &amp; Business band Radio</td>
<td>216 MHz to 470 MHz</td>
<td>15 kHz</td>
<td>±3 kHz</td>
<td>3 kHz</td>
</tr>
</tbody>
</table>
The 3 Important Parameters for FM and PM

The three important frequencies in FM and PM are

1. Carrier frequency $f_C$ (or $\omega_C$)
2. Maximum modulation frequency $f_m$ (or $\omega_m$), and
3. Peak frequency deviation $\Delta f$ (or $\Delta \omega$)

Two Definitions of importance:

1. Modulation index $\beta$

$$\beta = \frac{\Delta f}{B_m} = \frac{\Delta f}{f_m} = \frac{\Delta \omega}{\omega_m} = \frac{\Delta \omega}{2\pi B_m}$$

(can be a very large number)

2. Deviation ratio $D$

$$D = \frac{\Delta f}{f_C} = \frac{\Delta \omega}{\omega_C}$$

(always much less than unity)

Remember: For FM $\Delta \omega = k_f m_p$ & for PM $\Delta \omega = k_p m'_p$
FM Bandwidth and the Modulation Index $\beta$

A. Narrowband FM (NBFM) – $\beta << 1$ radian

$$B_{FM}^{NB} \approx 2B_m$$  where $B_m$ is the bandwidth of $m(t)$

B. Wideband FM (WBFM) – $\beta >> 1$ radian

$$B_{FM}^{WB} \approx 2(\beta + 1)B_m$$,  where  $\beta = \frac{\Delta f}{B_m} = \frac{\Delta f}{f_m} = \frac{\Delta \omega}{\omega_m}$

$\Delta f$ is the peak frequency deviation  $\Delta f = \max[k_f m(t)]$

$$B_{FM}^{WB} \approx 2(\Delta f + B_m) = 2(\beta + 1)B_m \quad \Leftarrow \text{Carson's Rule}$$

For PM we have analogous equation,

$$B_{PM}^{WB} \approx 2(\beta + 1)B_m$$
Example: Bandwidth of FM Signal

The message signal input to a modulator is $10 \cdot \cos(2\pi \times 10^4t)$. If frequency modulation with frequency deviation constant $k_f = 10^4 \pi$ is performed, find the bandwidth of the resulting FM signal.

Solution:

$$\beta = \frac{1}{2\pi} \frac{k_f A_m}{f_m} = \frac{10^4 \pi \times (10)}{2\pi \times 10^4} = 5$$

$$B_{FM} = 2(\beta + 1) f_m = 2(5 + 1) \cdot 10 \text{ kHz} = 120 \text{ kHz},$$

using Carson's rule to calculate bandwidth $B_{FM}$.
Example: Equal Bandwidth for FM & PM Signals

If phase modulation is performed using the message signal $10 \cdot \cos(2\pi \times 10^4 t)$ used in the previous slide, find the phase deviation constant $k_p$ giving the PM signal the same bandwidth, namely, 120 kHz.

**Solution:**

For both the FM and PM signals to have the same bandwidth, $\beta$ and $\Delta f$ must be the same. For FM, $\Delta \omega = k_f A_m$; but for PM, $\Delta \omega = k_p m_p'$. Expressing the message signal $m(t) = A_m \cdot \cos(\omega_m t)$ gives

$$m'(t) = \frac{d}{dt}(A_m \cos(\omega_m t)) = -\omega_m A_m \cdot \sin(\omega_m t) \quad \Rightarrow \quad m'_p = -\omega_m A_m$$

Thus,

$$k_f A_m = -k_p \omega_m A_m \quad \Rightarrow \quad k_p \frac{\omega_m}{\omega_m} = \frac{k_f}{\omega_m} = \frac{10^4 \pi}{2\pi \times 10^4} = \frac{1}{2} \quad \Leftarrow$$

Check: $\beta = \frac{k_p m'_p}{\omega_m} = \frac{k_p \omega_m A_m}{\omega_m} = k_p A_m = \frac{1}{2} (10) = 5$
Example: Commercial FM Radio Stations

For commercial FM radio, the audio message signal has a spectral range of 30 Hz to 15 kHz, and the FCC allows a frequency deviation of 75 kHz. Estimate the transmission bandwidth for commercial FM using Carson’s Rule.

Solution:

We start by calculating $\beta$

$$\beta = \frac{\Delta f}{B_m} = \frac{75 \text{ kHz}}{15 \text{ kHz}} = 5$$

Using Carson's rule gives

$$B_{FM} = 2(\beta + 1)B_m = 2(5 + 1)15 \text{ kHz} = 180 \text{ kHz}$$

The allowed bandwidth for commercial FM is 200 kHz. Note that Carson's rule slightly underestimates the bandwidth.
Why Does FM and PM Take Much More Bandwidth?

Observation: The bandwidth required for AM and NBFM are the same.

However, WBFM (wideband FM) requires much more bandwidth. Why?

A Fourier spectrum of an FM signal shows that to keep the amplitude constant of an FM signal that many components are required to represent the FM waveform. The frequency spectrum of an actual FM signal has components extending infinitely, although their amplitude decreases for sufficiently higher frequencies. Sufficiently higher frequencies applies to frequencies above the Carson bandwidth rule.

\[ B_{FM}^{WB} \approx 2(\Delta f + B_m) = 2(\beta + 1)B_m \quad \Leftarrow \quad \text{Carson's Rule} \]

Next we examine the Fourier components this using phasors.
Note: Only magnitudes of spectral lines shown.
NBFM Constructed From Phasors in FM Modulation

NBFM with tone modulation

\[ A_C \]

\[ f_c \]

\[ f_m \]

\[ \varphi(t) \]

\[ \frac{\beta A_C}{2} \]

\[ \frac{\beta A_C}{2} \]

\[ f_c - f_m \]

\[ f_c + f_m \]

\[ -\frac{\beta A_C}{2} \]
WBFM Phasor Diagram for Arbitrary $\beta$

Sidebands Constructed From Phasors in FM Modulation

Animation showing how phase modulation works in the phasor picture -- phase modulation with a sinusoidal modulation waveform and a modulation depth of \( \pi/4 \) radians. The blue line segments represent the phasors at the carrier and the harmonics of the modulation frequency.
Generating FM Signals

There are two basic methods to generate FM:

1. **Direct Method** (uses voltage-controlled oscillator to vary the frequency linearly with the message signal $m(t)$)
   
   **Advantage:** Can generate large frequency deviation.
   **Disadvantage:** Carrier frequency tends to drift and must be stabilized.

2. **Armstrong’s Indirect Method** (first generate NBFM with the message signal with a small frequency deviation and then frequency multiplication is used to increase the frequency and frequency deviation to desired levels (generates WBFM))
   
   **Advantage:** More stable carrier frequency.
   **Disadvantage:** More complex hardware and cost.
Direct Generation of FM Signal Using a VCO

VCO is “voltage-controlled oscillator”

$$\omega_{osc} \sim \frac{1}{\sqrt{LC_{eq}}}$$

$$C_{eq}$$ is capacitance $$C_D$$ plus capacitance of other capacitors.
Direct Generation of FM Signal Using a VCO and PLL

Input: \( m(t) \)

Crystal Oscillator

\( f_{\text{osc}} = \frac{f_C}{N} \)

Frequency Divider \( \div N \)

Mixer used as phase detector:

\[
\cos(\omega_c t) \cdot \cos(\omega_c t + \varphi) = \frac{1}{2} \left[ \cos(\omega_c t - \omega_c t - \varphi) + \cos(\omega_c t + \omega_c t + \varphi) \right] = \frac{1}{2} \left[ \cos(-\varphi) + \cos(2\omega_c t + \varphi) \right] = \frac{1}{2} \left[ \cos(\varphi) + \cos(2\omega_c t + \varphi) \right]
\]
A crystal filter is placed in the feedback loop to stabilize the oscillator. The frequency of oscillation can be pulled slightly from the high-Q crystal resonator’s frequency. The frequency deviates only slightly and is typically only up to about 100 ppm. However, the oscillator is very stable for $m(t) = 0$. 

A crystal is an electro-mechanical resonator.
A crystal oscillator is an electronic oscillator circuit that uses the mechanical resonance of a vibrating crystal of piezoelectric material to create an electrical signal with a precise frequency.

A major reason for the wide use of crystal oscillators is their high Q factor. A typical Q value for a quartz oscillator ranges from $10^4$ to $10^6$, compared to perhaps $10^2$ for an LC oscillator.

The maximum $Q$ for a high stability quartz oscillator can be estimated as $Q = 1.6 \times 10^7/f$, where $f$ is the resonant frequency in megahertz.

https://en.wikipedia.org/wiki/Crystal_oscillator
Generation of **Narrowband Frequency Modulation (NBFM)**

\[
\phi_{FM}(t) = A_C \cos \left( \omega_c t + k_f \int_{-\infty}^{t} m(\alpha) d\alpha \right)
\]

NBFM requires \( \beta \ll 1 \) radian

Agbo & Sadiku  
*Figure 4.5; page 168*
Indirect Generation of FM Using Frequency Multiplication

In this method, a narrowband frequency-modulated signal is first generated and then a frequency multiplier is used to increase the modulation index. The concept is shown below:

\[ m(t) \rightarrow \text{NBFM} \rightarrow \phi_{FM}^{NB}(t) \rightarrow \phi_{FM}^{WB}(t) \]

A frequency multiplier is used to increase both the carrier frequency and the modulation index by integer \( N \).
Frequency Multipliers

A frequency multiplier is a nonlinear component followed by a bandpass filter at the multiplied frequency desired.

We select the $n^{th}$ order nonlinear component of $y(t)$ and pass it through the bandpass filter.

\[ \varphi_{in}(t) = A_C \cdot \cos \left[ \omega_C t + k_f \int_0^t m(\lambda) d\lambda \right], \text{ and} \]

\[ \varphi_{out}(t) = A_C \cdot \cos \left[ n\omega_C t + nk_f \int_0^t m(\lambda) d\lambda \right] \]

**Note:** $m(t)$ is not distorted by multiplier.

**Conclusion:** Carrier frequency is now $nf_C$ and frequency deviation is now $n\Delta f$. Commercial frequency multipliers are generally $\times 2$ and $\times 3$. 

Section 4.4; Page 181 of Agbo & Sadiku
Armstrong Indirect FM Transmitter Example

Crystal stabilized **voltage-controlled oscillator**

These numbers correspond to an FM broadcast radio station.
Why are Two Multiplication Chains Used?

\[ \phi_{FM}^{NB}(t) \]

NBFM generator \rightarrow Multiplier Chain A \rightarrow Mixer \rightarrow Multiplier Chain B \rightarrow \phi_{FM}^{WB}(t)

Oscillator

[Image of a circuit diagram with labeled nodes and arrows indicating the flow of signals.]
Many Ways to Perform Frequency Multiplication

In electronics, a **frequency multiplier** is an electronic circuit that generates an output signal whose output frequency is a harmonic (multiple) of its input frequency. Frequency multipliers consist of a nonlinear circuit that distorts the input signal and consequently generates harmonics of the input signal.

Most multipliers are doublers or triplers
Frequency Multiplication Using Comb Generation

From our discussion on Fourier series and pulse trains:

Amplitude \( V \)

\[ T_p \]

\[ T \]

Comb frequencies shown

\[ X(f) \]

Envelope: \( \frac{T}{T_p} \text{sinc}(Tf) \)

\[ 0 \quad 1 \quad \frac{1}{T_p} \quad \frac{2}{T_p} \quad \rightarrow f(\text{Hz}) \]
Simple Comb Generator

A step recovery diode (SRD) is a p-n junction diode having the ability to generate extremely short pulses. It is also called snap-off diode or charge-storage diode, and has a variety of uses in microwave electronics (e.g., pulse generator or parametric amplifier).

https://www.edn.com/electronics-blogs/the-emc-blog/4402169/DIY-6-GHz-comb-generator
Step Recovery Diode Based Comb Generation

The key to generating a wide comb of frequencies is to generate very narrow pulses which step recovery diodes are designed to do.

Generation of Narrowband Phase Modulation (NBPM)

\[ \varphi_{PM}(t) = A_C \cos(\omega_c t + k_p m(t)) \]

Agbo & Sadiku
Figure 4.5; page 168
Generation of Narrow Band Phase Modulation

Carrier frequency $f_C$

$m(t)$ $k_p$ $C_D$ $\varphi_{PM}(t)$

Varactor diode

Limitation 1: Only a small amount of phase shift is generated (low-deviation)
Limitation 2: All phase-shift circuits produce amplitude variations.

https://www.slideshare.net/sghunio/chapter06-fm-circuits
Advantages of frequency modulation

1. **Resilient to noise:** The main advantage of frequency modulation is a reduction in noise. As most noise is amplitude based, this can be removed by running the received signal through a limiter so that only frequency variations remain.

2. **Resilient to signal strength variations:** In the same way that amplitude noise can be removed, so too can signal variations due to channel degradation because it does not suffer from amplitude variations as the signal level varies. This makes FM ideal for use in mobile applications where signal levels constantly vary.

3. **Does not require linear amplifiers in the transmitter:** As only frequency changes contain the information carried, amplifiers in the transmitter need not be linear.

4. **Enables greater efficiency:** The use of non-linear amplifiers (e.g., class C and class D/E amplifiers) means that transmitter efficiency levels can be higher. This results from linear amplifiers being inherently inefficient.
Disadvantages of FM

Disadvantages of frequency modulation

1. *Requires a more complicated demodulator*: One of the disadvantages is that the demodulator is a more complicated, and hence more expensive than the very simple diode detectors used in AM.

2. *Sidebands extend to infinity*: The sidebands for an FM transmission theoretically extend out to infinity. To limit the bandwidth of the transmission, filters are used, and these introduce some distortion of the signal.
Ideal FM Differentiator Demodulator

The ideal FM detector converts the FM signal’s instantaneous frequency $\omega_i$ to an amplitude that is proportional to $\omega_i$.

**Differentiation performs FM to AM conversion**

**Input:**

$$\phi_{FM}(t) = A_C \cos(\omega_C t + \theta(t)) = A_C \cos\left(\omega_C t + k_f \int_{-\infty}^{t} m(\lambda) d\lambda\right)$$

**Output:**

$$\frac{d\phi_{FM}(t)}{dt} = \frac{d}{dt}\left[A_C \cos\left(\omega_C t + k_f \int_{-\infty}^{t} m(\lambda) d\lambda\right)\right]$$

$$\frac{d\phi_{FM}(t)}{dt} = -A_C \left(\omega_C + k_f m(t)\right) \cdot \sin\left(\omega_C t + k_f \int_{-\infty}^{t} m(\lambda) d\lambda\right)$$

Both AM and FM included

After DC removal

AM allows the envelope detector to be used
Bandpass Limiter at the Receiver

For an envelope detector to work well the FM signal’s amplitude should be constant or flat. We can accomplish with a “hard limiter.” Factors such as channel noise, interference and channel fading result in amplitude variations in an FM signal’s amplitude at the receiver.

\[
x(t) = \phi_{FM}(t) = \frac{4}{\pi} \left[ \cos(\omega_C t) + k_f t \int_{-\infty}^{t} m(\lambda) d\lambda \right]
\]
Practical FM Differentiator Demodulator

Differentiator at low frequencies

\[ H(j\omega) = \frac{j\omega RC}{1 + j\omega RC} = \frac{j(\omega / \omega_{3dB})}{1 + j(\omega / \omega_{3dB})}; \quad \text{where} \quad \omega_{3dB} = \frac{1}{RC} \]

For \( \omega \ll \omega_{3dB} = \frac{1}{RC} \); then \( H(j\omega) \approx j\omega RC \)

Multiplication by \( j\omega \) in the frequency domain is equivalent to differentiation in time domain! The high-pass filter acts as a differentiator for an FM signal. Therefore,

\[ y(t) = A_c \omega_c RC + A_c \omega_c RCk_t m(t) \]

Envelope detector extracts \( m(t) \)
Bode Plot of CR High-Pass Filter

Bode Diagram

20 dB/decade

Magnitude [dB]

Frequency [rad/s]

Phase [deg]
Practical Frequency Demodulators

Frequency discriminators can be built in various ways:

• Time-delay demodulator
• FM slope detector
• Balanced discriminator
• Quadrature demodulators
• Phase locked loops (a superior technique)
• Zero crossing detector
This is an implementation of discrete time approximation to differentiation.

\[ y(t) = \frac{1}{\tau} \left( \phi_{FM}(t) - \phi_{FM}(t - \tau) \right) \]

\[ \frac{d\phi_{FM}(t)}{dt} = \lim_{\tau \to 0} \left[ y(t) \right] = \lim_{\tau \to 0} \left[ \frac{1}{\tau} \left( \phi_{FM}(t) - \phi_{FM}(t - \tau) \right) \right] \]

It can be shown that an adequate value for \( \tau \) is less than \( T/4 \), where \( T \) is the period of the unmodulated carrier for the FM signal. Again, this relies upon FM to AM conversion after which the envelope detector recovers \( m(t) \).
An FM Slope Detector Performs FM to AM Conversion

Comment: The differentiation operation is performed by any circuit acting as a frequency-to-amplitude converter.

Slope approximation

Operates on the skirt of the LC resonance curve
Balanced Discriminator (Foster-Seeley Discriminator) – 1936

Centered around $f_c$

Two tuned circuits

Envelope detectors

Another example of the use of symmetry in design.
Quadrature Demodulator – Block Diagram

FM signal is converted into PM signal

PM signal is used to recover the message signal \( m(t) \)

\( \varphi_{FM}(t) \)

Phase Shifting Circuit

Phase Comparator Circuit

Low-Pass Filter

Signal delay \( \tau_0 \) times carrier frequency \( f_c \) = 90 degrees (or \( \pi/2 \)).

Phase Detector
Using a XOR Gate for Phase Frequency Detector

- **Purpose:** To produce a signal current or voltage, proportional to the difference in phase or frequency between two input signals.

- **Example**

- **Logic Table**

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
Quadrature Demodulator – Implementation

The signal is split into two components. One passes through a network providing a basic 90° phase shift in addition to the phase shift from the signal’s frequency deviation. The mixer output is dependent upon the phase difference between the two signals; that is, it acts as a phase detector producing a voltage output proportional to the phase difference and thus the frequency deviation on the FM signal.

Phase-Locked Loops (Using Feedback)

A PLL consists of three basic components:
- Phase detector
- Loop filter
- Voltage-controlled oscillator (VCO)

PLL Diagram:

\[ A_C \left[ \cos \omega_C t + \theta_i(t) \right] \]

\[ 2A_{VCO} \left[ \cos \omega_C t + \theta_o(t) \right] \]

Output signal is phase

Bias Generator

Phase Detector

Low-Pass Filter

Oscillator (VCO)
Zero-Crossing Detectors

• Zero-Crossing Detectors are also used because of advances in digital integrated circuits.

• These are the frequency counters designed to measure the instantaneous frequency by the number of zero crossings.

• The rate of zero crossings is equal to the instantaneous frequency of the input signal.
Zero-Crossing Detector Illustration

Zero crossing detector

\[ \varphi_{FM}(t) \rightarrow \text{Hard Limiter} \rightarrow \text{Zero-crossing circuit} \rightarrow \text{Multi-vibrator} \rightarrow \text{Averaging circuit} \rightarrow m(t) \]

More frequent ZC’s gives higher instantaneous frequency which causes greater average signal.

https://www.slideshare.net/avocado1111/angle-modulation-35636989
Noise in Frequency Modulation

In FM systems noise has a greater effect on the higher modulating frequencies. It is common practice to boost the signal level of the higher modulating frequencies to improve the signal-to-noise ratio of the overall transmitted FM signal.

This artificial boosting at the transmitter is called “pre-emphasis” and the removal of the boost at the receiver is called “de-emphasis.”

The result is an improvement in the discernible quality of received FM signals.

\[ S_{N_o}(f) = \frac{N_o (2\pi f)^2}{A_C^2} \text{ for } |f| < \frac{B_T}{2} \]

Power Spectral Density (PSD) of output noise in an FM receiver.
(Increases because noise is differentiated in FM receiver)

Fig. 10.9; p. 578 of 4th ed., Lathi & Ding
Channel noise acts as interference in FM and is uniform over the entire BW. Voice and music have more energy at lower frequencies, so we need to “emphasize” their upper frequencies by filtering. However, the HF emphasis must be removed at the receiver using a “de-emphasis” filter.

(Widely used commercially in the recording industry)

Filtering improves SNR in FM transmission.
## Typical Pre-Emphasis and De-Emphasis Filters

<table>
<thead>
<tr>
<th>Transmitter</th>
<th>Receiver</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Pre-emphasis Filter</strong></td>
<td><strong>De-emphasis Filter</strong></td>
</tr>
</tbody>
</table>

![Pre-emphasis Filter Schematic](image)

\[
H(\omega) = \frac{V_{out}}{V_{in}} = \frac{1 + j\omega R_1 C}{1 + j\omega (R_1 R_2) C}
\]

\[
\left|H(\omega)\right| (dB) = \log_{10} \left( \frac{1}{R_1 C} \right) - 6 \, \text{dB/octave}
\]

2.1 kHz \quad 33 \, kHz

![De-emphasis Filter Schematic](image)

\[
H(\omega) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + j\omega R_1 C}
\]

\[
\left|H(\omega)\right| (dB) = \log_{10} \left( \frac{1}{R_1 C} \right) - 6 \, \text{dB/octave}
\]

2.1 kHz \quad \log(\omega)
Analog and Digital FM Cellular Telephones

1G analog cellular telephone (1983) – AMPS (Advanced Mobile Phone Service)
First use of cellular concept . . .
Used 30 kHz channel spacing (but voice BW was B = 3 kHz)
   Peak frequency deviation $\Delta f = 12$ kHz, and
   $B_T = 2(\Delta f + B) = 2(12$ kHz $+ 3$ kHz) $= 30$ kHz
Two channels (30 kHz each); one for uplink and one for downlink
Used FM for voice and FSK (next slide) for data communication
No protection from eavesdroppers!

Successor to AMPS was GSM (Global System for Mobile) in early 1990s
GSM is 2G cellular telephone
Still used by nearly 50% of world’s population (as of 2017)
GSM was a digital communication system
   Modulating signal is a bit stream representing voice signal
Uses Gaussian Minimum Shift Keying (GMSK)
Channel bandwidth is 200 kHz (simultaneously shared by 32 users
   This is 4.8 times improvement over AMPS

More to come on cellular . . .
Digital Carrier Modulation – ASK, FSK and PSK

$m(t)$

Amplitude Shift Keying

Frequency Shift Keying

Phase Shift Keying

$k_p m_d(t) \sim (-\pi, \pi)$

https://slideplayer.com/slide/12711804/
Binary Phase Shift Keying (BPSK)  

The wave shape is ‘symmetrical’ at each phase transition because the bit rate is a sub-multiple of the carrier frequency \( \omega_c/(2\pi) \). In addition, the message transitions are timed to occur at the zero-crossings of the carrier.
Questions?

Additional slides
Triangular-Wave FM Generation

\[ \varphi_{FM}(t) = \int m(t) \, dt \]

Schmitt Trigger

Inverter

Switch

Integrator

\[ v_{out} \]

\[ v_{in} \]

\[ v_{out} \]

\[ v_{in} \]

Able to generate FM up to 30 MHz

Switching-Circuit Phase Modulator

FM System Improvement in SNR

The signal-to-noise ratio (SNR) improvement in an FM system is a function of modulation index \( \beta \),

\[
SNR_{FM} = 3 \beta^3 (\beta + 1) \cdot CNR,
\]

where \( CNR \) is carrier-to-noise ratio

\[
SNR_{FM} = 3 \left( \frac{B_T}{2B_m} \right) \cdot CNR
\]

**Example:** For FM transmission bandwidth \( B_T \) of 200 kHz and a message bandwidth \( B_m \) of 15 kHz (\( \beta = 5.67 \)), the improvement in the SNR at the output of an FM receiver to have an FM gain of 27 dB above the CNR.

This is essentially a tradeoff between message signal quality (SNR) and FM transmission bandwidth. Thus, greater transmission bandwidth is the key to FM’s superior performance.