Problem 1  FM and PM Relationship  (10 points)

When we generate a “phase-modulated” signal modulated by the message signal \( m_p(t) \), we simultaneously produce a “frequency modulated” signal. Of course, the frequency modulated signal corresponds to a different modulated waveform where we represent the message signal by \( m_f(t) \). Write an equation stating the relationship between \( m_p(t) \) and \( m_f(t) \). [Hint: Equate the angles for both modulations.]

Solution:

For phase modulation the angle is
\[
\theta(t) = k_p m_p(t),
\]
and for frequency modulation we have
\[
\theta(t) = k_f \int_{-\infty}^{t} m_f(\lambda) d\lambda
\]
Equating both expressions gives
\[
m_p(t) = \frac{k_f}{k_p} \int_{-\infty}^{t} m_f(\lambda) d\lambda \quad \text{or} \quad m_f(t) = \frac{k_p}{k_f} \frac{dm_p(t)}{dt}
\]
(a) If frequency modulation is performed by the modulator with frequency deviation constant \( k_f = 4\pi \cdot 10^4 \); find the minimum and the maximum values of the instantaneous frequencies, specifically \( f_{i,\text{min}} \) and \( f_{i,\text{max}} \), respectively.

Solution:

\[
c(t) = 5 \cdot \cos(2\pi \times 10^7 t); \quad f_c = \frac{2\pi \cdot 10^7}{2\pi} = 10^7 \text{ Hz}
\]

\[
[m(t)]_{\text{max}} = 2; \quad [m(t)]_{\text{min}} = -2.
\]

\[
f_{i,\text{max}} = f_c + \frac{k_f}{2\pi} [m(t)]_{\text{max}} = 10^7 + \frac{4\pi \times 10^4}{2\pi} [2] = 10.04 \text{ MHz}
\]

\[
f_{i,\text{min}} = f_c + \frac{k_f}{2\pi} [m(t)]_{\text{min}} = 10^7 + \frac{4\pi \times 10^4}{2\pi} [-2] = 9.96 \text{ MHz}
\]

(b) Sketch the FM modulated waveform.
Problem 3 Frequency Deviation of an FM Signal (10 points)

Frequency modulation is performed with a tone message signal given by $m(t) = 2 \cdot \cos(\omega_m t)$, and a carrier given by $c(t) = 5 \cdot \cos(10 \omega_m t)$, where $\omega_m = 3\pi \cdot 10^4$ radians/sec.

(a) What is the maximum possible value of the frequency deviation constant $k_f$ if the modulated signal is narrowband FM? (8 points)

Solution:

$$m(t) = 2 \cos \omega_m t ; \quad \omega_m = 3\pi \cdot 10^4 ; \quad \beta = \frac{\Delta \omega}{\omega_m} = \frac{A_m k_f}{\omega_m} = \frac{2k_f}{3\pi \cdot 10^4}$$

But NBFM implies that $\beta < 0.3$. Take $\beta = 0.3$ as the boundary value.
\[ \because \quad \beta = \frac{2k_f}{3\pi \times 10^4} \leq 0.3 \quad \Rightarrow \quad k_f \leq 4.5\pi \times 10^3 \quad (=14,137.17) \]

(b) If \( k_f = 1.5\pi \times 10^3 \), obtain an expression for the spectrum of the narrowband FM signal and sketch its positive frequency amplitude spectrum. (7 points)

\[
\Phi_{FM}(\omega)
\]

Solution:

\[
k_f = 1.5\pi \times 10^3 \quad \beta = \frac{\Delta \omega}{\omega_m} = \frac{A_m k_f}{\omega_m} = \frac{2[1.5\pi \times 10^3]}{3\pi \times 10^4} = 0.1 < 0.3 \quad \Rightarrow \quad \text{NBFM}
\]

Employing Eq. (4.19) on page 169,

\[
\phi_{FM}(t) = A_c \cos(\omega_c t) + \frac{1}{2} \beta A_c \cos\left((\omega_c + \omega_m)t\right) \frac{1}{2} \beta A_c \cos\left((\omega_c - \omega_m)t\right)
\]

\[
5 \cos(\omega_c t) + 0.25 \cos\left((\omega_c + \omega_m)t\right) - 0.25 A_c \cos\left((\omega_c - \omega_m)t\right)
\]

\[
\Phi_{FM}(\omega) = 5\pi \left[ \delta(\omega + \omega_c) + \delta(\omega - \omega_c) \right] + 0.25\pi \left[ \delta(\omega + \omega_c + \omega_m) + \delta(\omega - \omega_c - \omega_m) \right] - 0.25\pi \left[ \delta(\omega + \omega_c - \omega_m) + \delta(\omega - \omega_c + \omega_m) \right]
\]

The spectrum is like that of Figure 4.6, except for the differences in sideband frequencies and amplitudes.
Problem 4  FM Bandwidth  (15 points)

Assume that the bandwidth of the message signal used in Problem 2 above is band-limited to the third harmonic of its fundamental frequency, and that the frequency modulation operation has a frequency deviation constant of \( k_f = \pi \cdot 10^5 \).

(a) Find the frequency deviation of the FM signal.  (4 points)

Solution:
The bandwidth of \( m(t) \) going up to the third harmonic is

\[
B = 3 \frac{1}{T} = 3 \times \frac{1}{0.1\text{ms}} = 30 \text{ kHz}
\]

\[
\Delta f = \frac{1}{2\pi} \left[ m_p k_f \right] = \frac{1}{2\pi} \left[ 2 \times 10^5 \pi \right] = 100 \text{ kHz}
\]

(b) Find the phase deviation constant such that if phase modulation is performed, the PM signal will have the same bandwidth as the FM signal.  (4 points)

Solution:
For PM, \( \Delta f = 100 \text{ kHz} \), also since FM and PM bandwidths are equal. Using the graph in Problem 2 above we can determine the derivative of \( m(t) \),

\[
\frac{dm_p}{dt} = \frac{\Delta m(t)}{\Delta t} = \frac{2 - 0}{\left( 0.05 \times 10^{-3} - 0 \right) \text{sec}} = 4 \times 10^4 \text{ s}^{-1}
\]
\[ \Delta f = 100 \text{ kHz} = \frac{k_p \times \left( \frac{dm_p}{dt} \right)}{2\pi} = \frac{k_p \times 4 \times 10^4}{2\pi} \Rightarrow k_p = 5\pi \]

Note: The Solution Manual for Agbo & Sadiku is in error (it is not 2.5 \( \pi\); don’t blindly believe a solution manual, especially when a book is in its first edition).

(c) Find the bandwidth of the FM signal using Carson’s rule. (4 points)

Solution:
Using Carson’s rule:
\[ B_{FM} = 2(\Delta f + B) = 2(100 \text{ kHz} + 30 \text{ kHz}) = 260 \text{ kHz} \]

(d) Find the bandwidth of the FM signal using the conservative rule given by equation (4.32) on page 180. (3 points)

Solution:
\[ B_{FM} = 2(\Delta f + 2B) = 2(100 \text{ kHz} + 60 \text{ kHz}) = 320 \text{ kHz} \]

Problem 5  FM Waveform  (15 points)

The figure below shows an FM carrier modulated by a single-tone sinusoidal wave. Calculate both the carrier frequency \( f_C \) and the frequency of the tone frequency \( f_m \). Express both frequencies in kilohertz (kHz).

Carrier frequency \( f_C = \) ________292.1____ kHZ

Tone frequency \( f_m = \) ________22.47____ kHZ

Approximately, it is hard to exactly read the values from the graph. Therefore, any an answer close to the above values is acceptable.
Answer:

For the tone frequency we identify one cycle of the FM signal frequency as being modulated at frequency $f_m$. This is shown below where thirteen cycles of the FM waveform occur before it repeats over and over (obviously it repeats at the reciprocal of the period $T_m$ of the modulating frequency $f_m$. The red arrows show the points picked to determine the period $T_m$. To determine the time for one cycle of the carrier simply divide the time between the arrows by the number of cycles occurring.
Number of cycles $= 13$ and $T_m = 67.5 - 23 \mu\text{sec} = 44.5 \mu\text{sec}$

The time for one carrier cycle is $T_C = \frac{44.5 \mu\text{sec}}{13} = 3.423 \mu\text{sec}$

$$f_C = \frac{1}{T_C} = \frac{1}{3.423 \times 10^{-6} \text{ sec}} \approx 292,141 \text{ Hz} \approx 292.14 \text{ kHz}$$

The tone modulating frequency $f_m$ is

$$f_m = \frac{1}{T_m} = \frac{1}{44.5 \times 10^{-6} \text{ sec}} \approx 22,472 \text{ Hz} \approx 22.47 \text{ kHz}$$

The carrier frequency is 13 times greater than the tone frequency and both frequencies you found must be consistent with that.

**Problem 6  Generation of FM Signal With Multiplier  (20 points)**

The simple circuit shown below is a frequency multiplier.

The input FM signal is

$$\phi(t) = A_c \cdot \sin \left[ 2\pi \times 10^7 t + \sin \left( 2\pi \times 10^4 t \right) \right]$$

The half-wave rectified output from the diode is represented by $x(t)$ and the output of the circuit is represented by $\phi_n(t)$. 

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8  Homework 7
We want \( \phi_n(t) \) to be a frequency multiplied signal where it is multiplied by integer \( n \) and \( \phi_n(t) \) has a carrier frequency equal to \( n f_C \). We do this by choosing inductor \( L \) and capacitor \( C \) having a series resonance equal to \( n f_C \). The frequency deviation for FM then sets the resistance value \( R \) in the circuit (we are ignoring the resistor at the cathode end of the diode).

Remember in a series resonant circuit has a resonance frequency \( f_{res} \) of

\[
f_{res} = \sqrt{\frac{1}{(2\pi)^2 LC}}.
\]

The half-power points of the resonance bell-shaped curve are the -3 dB points. Solving the equations for the \( LRC \) resonator gives the high and low frequencies corresponding to the half-power points as

\[
\omega_{H and L} = \pm \left( \frac{R}{2L} \right) + \sqrt{\left( \frac{R}{2L} \right)^2 + \frac{1}{LC}} ,
\]

where subscript “\( H \)” stands for the half-power frequencies above \( f_{res} \) and “\( L \)” stands for the half-power frequency below \( f_{res} \). If you like to work with the circuit \( Q \) possess, then we have \( B = f_H - f_L \)

\[
Q = \frac{f_{res}}{f_H - f_L} = \frac{\omega_{res}}{\omega_H - \omega_L} ; \quad Q = \frac{\omega_{res}L}{R} = \frac{1}{\omega_{res}CR} = \frac{1}{R \sqrt{C}} .
\]

Complete the design of this multiplier circuit by choosing the values of \( R \) and \( C \), given that \( L = 5 \) \( \mu \)H (microhenries), so that a multiplication factor \( n = 4 \), and the -3 dB frequencies of the series resonant circuit are one frequency deviation of the output FM signal above and below \( f_{res} \).

https://www.electronics-tutorials.ws/accircuits/series-resonance.html

Solution: (This is Problem 4.15 in the textbook Agbo & Sadiku)

In FM the message signal is found from the relationship,

\[
\phi(t) = A_c \sin \left[ 2\pi \times 10^7 t + \sin \left( 2\pi \times 10^4 t \right) \right] \Rightarrow \sin \left( 2\pi \times 10^4 t \right) = k_f \int_0^t m(\lambda) \, d\lambda
\]

\[
\therefore \frac{d}{dt} \left[ \sin \left( 2\pi \times 10^4 t \right) \right] = 2\pi \times 10^4 \cos \left( 2\pi \times 10^4 t \right) = k_f m(t) = k_f A_m \cos \omega_m t
\]
\[ \Delta \omega = 2\pi \Delta f = k_f A_m = 2\pi \times 10^4 \text{ rad/sec}; \quad \omega_c = 2\pi \times 10^7 \text{ rad/sec} \]

After multiplication by factor \( n = 4 \), the desired filter series resonant frequency and -3 dB frequencies are

\[ \omega_{res} = 4\omega_c = 8\pi \times 10^7 \]

\[ \omega_L = \omega_0 - 4\Delta \omega = 7.92\pi \times 10^7 \text{ rad/sec}; \]

\[ \omega_H = \omega_0 + 4\Delta \omega = 8.08\pi \times 10^7 \text{ rad/sec} \]

\[ \omega_{res} = 8\pi \times 10^7 = \frac{1}{\sqrt{LC}} \quad \Rightarrow \quad C = \frac{1}{\left(8\pi \times 10^7\right)^2 \times 5 \times 10^{-6}} = 3.17 \text{ pF} \]

or \( C = 3.17 \times 10^{-12} \text{ F} \)

It is not 0.317 nF as many students found because of the wrong value of \( f_c \) (and the Agbo & Sadiku Solution Manual is in error for anyone that might have access to it). The bandwidth \( B \) is

\[ B = \omega_H - \omega_L = 16\pi \times 10^4 = \frac{R}{L} \]

\[ \therefore \quad R = L\left(16\pi \times 10^4\right) = 5 \times 10^{-6} \left(16\pi \times 10^4\right) = 2.51 \Omega \]

**Problem 7 Differentiator Demodulator** (10 points)

The circuit shown below is a good approximation to an ideal differentiator demodulator if \( f_c < 0.1 f_H \), where \( f_c \) is the carrier frequency of the FM signal and \( f_H \) is the cut-off frequency of the RC high-pass filter. The input FM signal is given by

\[ \phi_{FM}(t) = 4 \cdot \cos\left[\pi \times 10^8 t + \pi \times 10^4 \left(5 \cdot \sin(\pi \times 10^4 t)\right)\right] \]

Specify the value of the resistance \( R \) such that \( f_c = 0.1 f_H \) if \( C = 5 \text{ pF} \).
Solution:
\[ \phi_{FM}(t) = 4 \cos \left( 10^8 \pi t + 10^4 \pi 5 \sin 10^4 \pi t \right) \Rightarrow \omega_c = 10^8 \pi; \ f_c = 50 \text{ MHz}. \]

The high-pass filter cut-off frequency is
\[ f_H = 10 f_c = 5 \times 10^8 \text{ Hz}. \]

\[ \omega_H = \frac{1}{RC} = 2 \pi f_H = 10^9 \pi \text{ rad/sec} \]

\[ R = \frac{1}{\omega_H C} = \frac{1}{10^9 \pi \times 5 \times 10^{-12}} = 63.7 \ \Omega \]