Problem 1 Balanced Discriminator (40 points)

Consider the balanced discriminator (Figure 4.19 on page 199) for demodulating a PM signal shown below:

![Balanced Discriminator Diagram]

The PM signal with bandwidth $B_{PM}$ is written as

$$\phi_{PM}(t) = 5 \sin \left[ (2\pi \times 10^7) t + 4 \cos(2\pi \times 10^4)t \right]$$

The upper and the lower slope detectors each have a bandwidth of $2B_{PM}$, and their resonant frequencies are $1.5B_{PM}$ above and below the PM carrier frequency, respectively. Complete the design of the balanced slope detector by calculating the values of $C_1$, $C_2$, $R_1$ and $R_2$, given $L_1 = L_2 = 4$ microhenries ($\mu$H).

Solution:

$$\phi_{PM}(t) = 5 \sin \left[ 2\pi \times 10^7 t + 4 \cos(2\pi \times 10^4)t \right] = 5 \sin \left[ 2\pi \times 10^7 t + k_p m(t) \right]$$

$$k_p m(t) = 4 \cos(2\pi \times 10^4 t) \Rightarrow k_p m'(t) = -8\pi \times 10^4 \sin(2\pi \times 10^4 t)$$

$$\Delta f = \frac{1}{2\pi} k_p m' = \frac{8\pi \times 10^4}{2\pi} = 40 \text{ kHz}; \ f_m = 10 \text{ kHz}; \ f_c = 10 \text{ MHz}$$

$$B_{FM} = 2(\Delta f + f_m) = 2(40 + 10) \text{ kHz} = 100 \text{ kHz}$$

$$\omega_o = \frac{1}{\sqrt{LC}}$$
\[ Q = \frac{\omega_0}{BW} = \omega_0 CR = \frac{R}{\omega_0 L}, \text{the frequency response is} \]
\[ H(j\omega) = \frac{1}{1 + j(\omega C - \frac{1}{\omega_0 L})} = \frac{1}{1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}. \]

For the upper slope circuit, let the resonant frequency be \( f_1 \).
\[ f_1 = f_c + 1.5B_{FM} = \left[10^7 + 1.5 \times 10^5 \right] \text{ Hz} = 10.15 \text{ MHz} \]
\[ C_1 = \frac{1}{\omega_1^2 L} = \frac{1}{(2\pi \times 10.15 \times 10^6)^2 \times 4 \times 10^{-6}} = 61.47 \text{ pF} \]
\[ R_1 = \frac{\omega_1^2 L_1}{BW} = \frac{(2\pi f_1)^2 L_1}{2\pi \times 2B_{FM}} = \frac{2\pi f_1^2 L_1}{2B_{FM}} = \frac{2\pi (10.15 \times 10^6)^2 \times 4 \times 10^{-6}}{2 \times 10^5} = 12.5 \text{ k}\Omega \]

For the lower slope circuit, let the resonant frequency be \( f_2 \).
\[ f_2 = f_c - 1.5B_{FM} = \left[10^7 - 1.5 \times 10^5 \right] \text{ Hz} = 9.85 \text{ MHz} \]
\[ C_2 = \frac{1}{\omega_2^2 L} = \frac{1}{(2\pi \times 9.85 \times 10^6)^2 \times 4 \times 10^{-6}} = 65.27 \text{ pF} \]
\[ R_2 = \frac{\omega_2^2 L_2}{BW} = \frac{(2\pi f_2)^2 L_2}{2\pi \times 2B_{FM}} = \frac{2\pi f_2^2 L_2}{2B_{FM}} = \frac{2\pi (9.85 \times 10^6)^2 \times 4 \times 10^{-6}}{2 \times 10^5} = 12.2 \text{ k}\Omega \]

Problem 2 Zero Crossings in PM and FM (30 points)

Consider a modulating wave \( m(t) \) that increases linearly with time \( t \), starting at \( t = 0 \). Mathematically this is expressed as
\[ m(t) = \begin{cases} at, & \text{for } t \geq 0 \\ 0, & \text{for } t < 0 \end{cases} \]

where constant \( a \) is the slope parameter of the ramp (see figure below). In this problem we study the zero-crossings of the both PM and FM waves produced by message signal \( m(t) \) given the following parameters:
\[ f_c = \frac{1}{4} \text{ Hz} \]
\[ a = 1 \text{ volt/sec} \]

(a) **Phase Modulation** \((k_p = \pi/2 \text{ radians/volt} \text{ and } A_C = 1 \text{ volt})\)

\[ \varphi_{PM}(t) = \begin{cases} 
A_c \cos(2\pi f_c t + k_p at), & \text{for } t \geq 0 \\
A_c \cos(2\pi f_c t), & \text{for } t < 0 
\end{cases} \]

When the angle in the argument of the cosine is an odd multiple of \(\pi/2\) the phase modulated wave \(\varphi_{PM}(t)\) experiences a zero crossing. Let \(t_n\) represent the instant in time \((n \text{ is an integer})\) for which \(\varphi_{PM}(t)\) exhibits a zero crossing event. Show that \((i.e., \text{ derive})\) the expression for \(t_n\) is given by

\[ t_n = \frac{\frac{1}{2} + n}{2f_c + \frac{k_p}{\pi} a} \]

and that this expression reduces to \(t_n = \frac{1}{2} + n\) applying the above parametric values for \(f_c, k_p\) and \(a\).

**Solution:**

Zero crossings occur for the condition that the argument of the cosine equal \((\pi/2 + n\pi)\), for \(n = 0, 1, 2, 3, \ldots\). We write the equation,

\[ 2\pi f_c t_n + k_p at_n = \frac{\pi}{2} + n\pi; \quad n = 0, 1, 2, \ldots \]

Solving for \(t_n\) gives

\[ t_n = \frac{\frac{1}{2} + n}{2f_c + \frac{k_p}{\pi} a} \]

and plugging in parameter values reduces the expression to
\[ t_n = \frac{\frac{1}{2} + n}{2f_c + \frac{k_p}{\pi} a} = \frac{\frac{1}{2} + n}{\frac{1}{2} + \frac{n}{4} + \frac{\pi/2}{\pi}} = \frac{\frac{1}{2} + n}{\frac{1}{2} + \frac{1}{2}} = \frac{1}{2} + n \quad \Leftarrow \]

for \( n = 0, 1, 2, 3, \ldots \).

(b) **Frequency Modulation** \((k_f = 1 \text{ hertz/volt and } A_C = 1 \text{ volt})\)

\[
\phi_{FM}(t) = \begin{cases} 
A_c \cos \left( 2\pi f_c t + \pi k_f a t^2 \right), & \text{for } t \geq 0 \\
A_c \cos \left( 2\pi f_c t \right), & \text{for } t < 0
\end{cases}
\]

Derive an expression for \( t_n \) for the FM case.

**Solution:**

Zero crossings occur for the condition that the argument of the cosine equal \((\pi/2 + n \pi)\), for \( n = 0, 1, 2, 3, \ldots \). We can write

\[
2\pi f_c t_n + \pi k_f a \left( t_n \right)^2 = \frac{\pi}{2} + n\pi; \quad n = 0, 1, 2, \ldots
\]

\[
-\frac{(\frac{1}{2} + n)\pi}{\pi k_f a} + \frac{2\pi f_c}{\pi k_f a} t_n + t_n^2 = 0
\]

Using the quadratic formula and taking only the positive square root term,

\[
t_n = \frac{-2f_c + \sqrt{\left(\frac{2f_c}{k_f a}\right)^2 - 4\left(\frac{-(\frac{1}{2} + n)}{(k_f a)^2}\right)}}{2}
\]

\[
t_n = \frac{1}{k_f a} \left[ -f_c + \sqrt{f_c^2 + k_f a \left(\frac{1}{2} + n\right)} \right]; \quad n = 0, 1, 2, \ldots \quad \Leftarrow
\]

Substituting,

\[
t_n = \frac{1}{4} \left[ -1 + \sqrt{9 + 8n} \right]; \quad n = 0, 1, 2, \ldots \quad \Leftarrow
\]
(c) On the graph below sketch the general behavior of both $\varphi_{PM}(t)$ and $\varphi_{FM}(t)$ for $t < 0$ and $t \geq 0$. Draw conclusions.

Solution:
Your plot should look something like this.

**General comments:**

(1) For phase modulation, the ramp adds phase to the argument of the cosine so the frequency for $t > 0$ appears to be a higher frequency but is constant because of the linear increase in phase.

(2) For frequency modulation, the linear increase in $m(t)$ appears to be a continuously increasing frequency for $t > 0$.

**Problem 3  FM Modulator**  (30 points)

The output of an FM modulator is given by
\[ \varphi_{FM}(t) = 20 \cdot \cos \left( \omega_c t + k_f \int_0^t m(\lambda) d\lambda \right) \]

where \( k_f = 20\pi \) radians/volt and \( m(t) \) is as shown below.

\( \varphi_{FM}(t) = A_C \cos \left( \omega_c t + \phi(t) \right) \),

where \( \phi(t) \) is the phase deviation. Thus,

\[ \phi(t) = 20\pi \int_0^t m(\lambda) d\lambda = \begin{cases} 0, & t < 0 \\ 100\pi, & 0 \leq t \leq 1 \\ 100\pi, & t > 1 \end{cases} \]

The sketch for \( \phi(t) \) is as shown below:

(a) Find the phase deviation \( \phi(t) \) as a function of time and sketch its shape.

**Solution:**

The output is of the general form,

\[ \varphi_{FM}(t) = A_C \cos \left( \omega_c t + \phi(t) \right) \],

where \( \phi(t) \) is the phase deviation. Thus,

\[ \phi(t) = 20\pi \int_0^t m(\lambda) d\lambda = \begin{cases} 0, & t < 0 \\ 100\pi, & 0 \leq t \leq 1 \\ 100\pi, & t > 1 \end{cases} \]

The sketch for \( \phi(t) \) is as shown below:
(b) Find the frequency deviation $\Delta f$ as a function of time and sketch its shape.

**Solution:**

To determine $\Delta f$ in hertz, we differentiate the phase deviation $\phi(t)$ and divide by $2\pi$, henceforth

$$\Delta f = \frac{1}{2\pi} \frac{d\phi(t)}{dt} = k_j m(t)$$

$$\Delta f = \frac{100\pi}{2\pi} = 50 \text{ Hz} \quad \text{for} \quad 0 \leq t \leq 1 \text{ sec} \quad \Leftarrow$$

and it is 0 otherwise

Sketch:

(c) Find the peak frequency deviation value in hertz.

**Solution:**

From the figure in part (b) immediately above it is obvious that the peak frequency deviation value is 50 Hz. \Leftarrow

(d) Find the peak phase deviation value in radians.

**Solution:**

From the figure in part (a) the peak phase deviation occurs at $t = 1$ second and is $100\pi$ radians. \Leftarrow
(e) What is the modulator’s output power? Assume into a one ohm resistance.

Solution:

\[ P = \frac{(20 \text{ V})^2}{2(1 \Omega)} = 200 \text{ watts} \]