ES 442 Homework #9 Solutions
(Spring 2020 – Due April 24, 2020)
Print out homework and do work on the printed pages.

Problem 1 Nyquist Sampling Rate & Interval (15 points)

From the Sampling Theorem, the Nyquist sampling rate must be twice the highest bandwidth of the message signal, $m(t)$. For the three message signals below determine the Nyquist rate and the Nyquist interval for each. The Nyquist sampling interval is equal to the reciprocal of twice the highest frequency component of the sampled signal.

(a) $m(t) = 3 \times \cos(50\pi t) + 10 \times \sin(300\pi t) + 1.5 \times \cos(120\pi t)$

**Solution:** The angles are stated in radians, so the three frequencies are 25 Hz, 150 Hz and 60 Hz, respectively. The highest frequency is 150 Hz, so the **Nyquist rate** is twice 150 Hz = 300 Hz. $\leftarrow$

The Nyquist interval is the reciprocal of the Nyquist rate = $1/300$ sec = 3.333 ms $\leftarrow$

(b) $m(t) = 8 \times \sin(600\pi t) \times \cos(200\pi t)$

**Solution:** We begin by using the identity $2 \sin(A) \times \cos(B) = \sin(A + B) + \sin(A - B)$. Therefore, $m(t) = 4 \times [\sin(600\pi t + 200\pi t) + \sin(600\pi t - 200\pi t)] = 4 \times [\sin(800\pi t) + \sin(400\pi t)]$. The highest frequency is 400 Hz; **Nyquist rate** is $2 \times 400$ Hz = 800 Hz. $\leftarrow$

The Nyquist interval is the reciprocal of the Nyquist rate = $1/800$ sec = 1.250 ms $\leftarrow$

(c) $m(t) = \text{sinc}(100t) = \frac{\sin(100\pi t)}{\pi t}$

Hint: Fourier Transform of a sinc waveform is a rectangular pulse,

$$g(t) = \text{sinc}(2Bt) \quad \xrightarrow{FT} \quad G(f) = \frac{1}{2B} \text{rect}\left(\frac{f}{2B}\right)$$

where "rect" is the symbol for a rectangular spectrum that is flat from frequency -B Hz to +B Hz. It is said to have a spectral bandwidth of B Hz.

**Solution:** Waveform $g(t)$ is unlimited in time, so it has a hard bandlimited frequency response. We have a bandwidth of 50 Hz, so the Nyquist rate is 100 Hz. $\leftarrow$

The Nyquist interval is the reciprocal of the Nyquist rate = $1/100$ sec = 10 ms $\leftarrow$

Note: Therefore, $2B = 100$ Hz, so $B = 50$ Hz.
Problem 2  ASCII Code (15 points)

The **American Standard Code for Information Interexchange** (ASCII) has 128 binary-coded characters. ASCII codes represent text in computers, telecommunications equipment, and other devices. If a computer generates 170,000 characters per second, determine the following:

(a) The number of bits (binary digits) required per character.
   
   For 128 levels we have \( L = 2^n \), where \( n = \) number of bits in PCM symbol.  
   
   We must represent 128 characters which corresponds to \( 2^n = 128 \). \( \therefore n = 7 \) bits.  

(b) The number of bits per second required to transmit the computer output, and the minimum bandwidth required to transmit this signal.
   
   We are told that we must transmit 170,000 characters per second.  
   
   For 7 bits per symbol, the bit rate will be  
   
   \[
   \text{Bit rate} = 7 \frac{\text{bits}}{\text{character}} \times 170,000 \frac{\text{character}}{\text{second}}
   \]
   
   \[
   \text{Bit rate} = 1,190,000 = 1.19 \times 10^6 \text{ bps (bits/second)} \]
   
   But the bandwidth is one-half the bit rate, therefore,  
   
   \[
   B_r = 595,000 \text{ Hz (note the units are hertz)} \]

(c) For single error detection capability, an additional bit (called a parity bit) is added to the code for each character. Modify your answers for part (a) and (b) to accommodate the addition of the parity bit to each character.
   
   Now we add another bit (i.e., the parity bit) so we require \( n = 8 \) bits per character.  
   
   This increases the bit rate to 1,360,000 bps (= 1.36 Mb/second)  
   
   Bandwidth calculation:  
   
   Therefore, \( B_r = 680,000 \text{ Hz} \)

Problem 3  CD Audio (20 points)
A compact disc (CD) records audio signals digitally using PCM. The audio baseband signal’s bandwidth is 15 kHz.

(a) If the Nyquist samples are uniformly quantized into \( L = 65,536 \) levels and then binary-coded; determine the number of binary digits (bits) \( n \) required to encode a sample. How many bits per second are being transmitted?

For 15 kHz as the upper frequency of signal \( m(t) \), the Nyquist sampling rate = \( 2 \times 15 \text{ kHz} = 30 \text{ KHz} \). This means we must take 30,000 samples/second. We have 65,536 levels which corresponds to 16 bits of resolution (We get this from \( 65,536 = 2^n \); so solving for \( n \) gives \( n = 16 \) bits. 

The transmission rate is 16 bits/sample \( \times \) 30,000 samples/second = 480,000 bits/second

(b) If the audio signal has an average power of 0.125 watt and a peak voltage of 1 volt. Find the resulting ratio of signal-to-quantization noise ratio (SQNR) of the uniform quantizer output in part (a).

Given the signal power to be \( P_m = 0.125 \) watt and \( m_p = 1 \) volt.

Using the equation for signal-to-quantization noise ratio,

\[
\text{SQNR} = \frac{P_m}{N_q} = 3L^2 \left( \frac{\langle m^2 \rangle}{m_p^2} \right) = 3L^2 \left( \frac{P_m}{m_p^2} \right) = 3(65,536)^2 \left( \frac{0.125}{(1)^2} \right) = 1.61 \times 10^9
\]

\[
\text{SQNR}_{dB} = 10 \cdot \log_{10} \left( 1.61 \times 10^9 \right) = 92.07 \text{ dB}
\]

(c) Determine the number of binary digits per second (bps) required to encode and transmit the audio signal.

Given the sampling rate to be 30,000 samples/second and with 16 bits, the bit rate becomes

\[
\text{Bit rate} = f_{\text{Nyquist}} \times N_{\text{bits}} = 30,000 \text{ Hz} \times 16 \text{ bits} = 480,000 \text{ bps}
\]

(d) In practice, CD music is sampled at a rate of 44,100 samples/second, which is above the Nyquist rate. Still retaining the number of levels \( L \) to be 65,536 levels,
determine the bit rate needed to support this higher sampling rate. What is the minimum bandwidth needed to support this higher data rate.

\[ \text{Bit rate} = f_{\text{Nyquist}} \times N_{\text{bits}} = 44,100 \, \text{Hz} \times 16 \, \text{bits} = 705,600 \, \text{bps} \]

Thus, the minimum bandwidth \( B_r \) is one-half the sampling rate

\[ B_r = \frac{1}{2} \left( f_{\text{Nyquist}} \times N_{\text{bits}} \right) = \frac{705,600}{2} \, \text{Hz} = 352,800 \, \text{Hz} \]

**Problem 4  Bit Rate and Baud Rate** (10 points)

A multi-level digital communication system sends one of 32 possible levels over a channel every 1.8 milliseconds.

(a) What is number of bits corresponding to each level?

32 levels corresponds to \( 2^5 \) levels, so the number of bits = 5 bits

(b) What is the Baud rate?

The Baud rate corresponds to the number of symbols that can be sent per unit time. If the symbol period is 1.8 milliseconds, the symbol frequency is

\[ f = \frac{1}{1.8 \times 10^{-3}} = 555.6 \, \text{Hz} \]

(c) What is the bit rate?

The bit rate corresponds to the number of bits that can be sent per unit time. If 556 symbols can be sent each second, and knowing there are 5 bits/symbol, thus, \( 5 \times 556 \, \text{bps} = 2,780 \, \text{bps} \)

**Problem 5  Delta Modulation** (10 points)

Consider a sinusoidal signal \( m(t) = A \cos(\omega_m t) \) where \( 2\pi f_m = \omega_m \), that is applied to a delta modulator with step size \( \Delta \). Show that slope overload distortion will occur when

\[ A > \frac{\Delta}{\omega_m T_s} = \frac{\Delta}{2\pi} \left( \frac{f_s}{f_m} \right) \]
where \( f_s = 1/T_s \) is the sampling frequency and \( T_s \) is the period between samples.

Starting with \( m(t) = A \cdot \cos(\omega_m t) \); then \( \frac{dm(t)}{dt} = -A \cdot \omega_m \sin(\omega_m t) \)

or \( \left| \frac{dm(t)}{dt} \right|_{\text{max}} = A \cdot \omega_m \) because the maximum value of \( \sin(\omega_m t) \) is unity.

When \( \frac{\Delta}{T_s} \geq A \cdot \omega_m \) equals or exceeds the maximum slope of \( m(t) \), slope overload does not occur. Slope overload occurs when below this limit. Therefore, if \( A > \frac{\Delta}{(\omega_m T_s)} \), slope overload will occur.

**Problem 6 Delta Modulation** (10 points)

A delta modulation system is designed to operate at 5 times the Nyquist rate for a signal with a 3 kHz bandwidth. The quantization step size is 250 mv. Find the maximum amplitude of a 1 kHz input sinusoidal for which the delta modulator does not show slope overload.

The threshold for slope overload is \( A = \frac{\Delta}{(\omega_m T_s)} = \frac{\Delta}{\omega_m} \cdot f_s \)

For this problem the Nyquist sampling rate \( f_s \) is 5 times \( 2B \), where \( B \) is the bandwidth 3 kHz; hence, \( f_s = 5 \times 2 \times 3 \, \text{kHz} = 30 \, \text{kHz} \).

The signal frequency \( \omega_m \) is \( 2\pi \times 1 \, \text{kHz} = 6,283 \, \text{radian/sec} \).

Maximum amplitude \( A = \frac{\Delta}{\omega_m} \cdot f_s = \frac{0.25}{6283} \times 30,000 = 1.194 \, \text{volt} \)

**Problem 7 Delta Modulation** (20 points)

For this problem you have a delta modulation (DM) system for transmitting voice signals. Assume the sampling frequency is 256 kHz and the voice bandwidth 3 kHz. The system is designed to have a maximum signal amplitude of 10 volts. Find the following parameters:
(a) What is the minimum quantization step size $\Delta$ allowed for this DM system?

$$\omega_m = 2\pi \times 3000 \text{ rad/sec} = 18,850 \text{ rad/sec}; \quad A_m = 10 \text{ volts}$$

$$f_s = 256 \text{ kHz} \quad \text{implies} \quad T_s = 3.906 \times 10^{-6} \text{ second}$$

$$\Delta_{\text{min}} = \omega_m A_m T_s = 18,850 \times 10 \times 3.906 \times 10^{-6} = 0.736 \text{ volt} \quad \Leftarrow$$

(b) Determine the average power of the granular noise.

We know that the granular noise power is given by $N_q \approx \frac{\Delta^2}{3}$

Therefore, $N_q \approx \frac{(0.736)^2}{3} = \frac{0.542}{3} = 0.181 \text{ watt} \quad \Leftarrow$

Remember this is normalized to a resistance of one ohm.

(c) Determine the minimum channel bandwidth (in bits per second) required to transmit this DM signal.

Finally we determine the channel bandwidth required.

Transmission bandwidth $BW = 1 \frac{\text{bit}}{\text{sample}} \times 256,000 \frac{\text{samples}}{\text{second}}$

Transmission bandwidth $BW = 256,000 \frac{\text{bits}}{\text{second}} \quad \Leftarrow$

But bandwidth $B_T$ is one-half of this value, therefore,

$$B_T = 128 \text{ kHz} \quad \Leftarrow$$