The Fourier Transform

EE 442 Analog & Digital Communication Systems

Lecture 4

Voice signal

time

frequency (Hz)

Sampled Sound Data
FFT Magnitude
Summary of Lecture 3 – Page 1

For a linear time-invariant network, given input \( x(t) \), the output \( y(t) = x(t) * h(t) \), where \( h(t) \) is the unit impulse response of the network in the time domain. In the frequency domain, the output spectral response is \( Y(f) = X(f) \cdot H(f) \), where \( Y(f) \), \( X(f) \) and \( H(f) \) are the Fourier transforms of \( y(t) \), \( x(t) \) and \( h(t) \), respectively.

For a **distortionless** channel or network, we require the relationship of output to input to be \( y(t) = K \cdot x(t - t_d) \), where \( x(t) \) is the input, \( K \) is a constant and \( t_d \) is a time delay.

The **time delay** \( t_d(f) \) for a two-port network \( N \) is given by \( t_d(f) = -\frac{\phi_N(f)}{2\pi f} \).

For a transmission line (e.g., coax) with constant wave velocity the delay time is \( t_d(f) = \frac{\text{line length}}{\text{velocity}} \).

An ideal filter (with infinitely steep edges or “brick wall” cutoff) is not realizable because its unit impulse response would have to be non-causal (like a sinc function in time).

All practical filters (with a finite number of non-deal components) have sloped edges meaning that higher frequencies of the input signal are still transmitted but with significant attenuation.

Commonly used filters include the **Butterworth** filter (aka maximally flat magnitude in its passband), the **Chebychev** filter (aka “equal-ripple”) and the **Bessel-Thompson** filter (aka maximally flat phase delay). Of these three filters the Chebychev gives the sharpest cutoff response.
**Summary of Lecture 3 – Page 2**

**Group delay** is defined as $t_{gr}(f) = -\frac{d\phi_N(f)}{d(2\pi f)}$ and gives the delay of the energy transport of the signal.

Group delay is sometimes called the **envelope delay** of a network or transmission line.

Group delay is (1) a measure of a network’s phase distortion, (2) the transit time of signal’s envelope through the network versus frequency, and (3) the derivative of the phase characteristic with respect the frequency (the mathematical interpretation).

Group delay variation causes distortion of the signal waveform as it passes through a network, or travels over a channel.

Phase delay is the time delay experienced by the carrier wave as it passes through the network and group delay is the time delay experienced by the envelope (signal energy) as it passes through the network.

Amplitude vs. frequency distortion typically flattens the pulse shape, whereas phase vs. frequency distortion typically skews the pulse shape.
Summary of Lecture 3 – Page 3

Wireless Signal Transmission:
The first radio transmissions were in the 1890’s and Marconi was a primary player in this work.

The free space loss for electromagnetic waves spreading from a point source is

\[
\text{FSPL} = \left(\frac{4\pi r}{\lambda}\right)^2 = \left(\frac{4\pi f}{c}\right)^2 \quad \text{where} \quad c = \lambda f
\]

\[
\text{FSPL (dB)} = 10 \cdot \log_{10} \left(\left(\frac{4\pi f}{c}\right)^2\right)
\]

The Friis’ loss formula for antenna-to-antenna loss is given by

\[
P_r = P_i \left( G_i \cdot G_r \right) \left(\frac{\lambda}{4\pi r}\right)^2
\]

Radio wave propagation in the atmosphere: (1) space-wave propagation (e.g., satellite-to-ground), (2) sky-wave propagation (bounce EM waves off ionosphere), (3) line-of-sight propagation, and (4) ground-wave propagation (EM wave follows the Earth’s contour because of diffraction – limited to less than 2 MHz frequencies and it is very lossy).

Causes of deterioration of wireless signals – Free-space loss, multipath signals combine at receiver, shadowing, mobility (Doppler shifting), interference, noise and the channel characteristics is time-varying.

Fading is divided into large-scale fading (path loss and shadowing) and small-scale fading (multipath and Rayleigh fading ⩾ λ/2).
A simplified path-loss model is \( P_r = P_t K \left( \frac{d_0}{D} \right)^\gamma \)

where \( K \) is a path-loss constant, \( d_0 \) is the distance from the antenna to the far field region, \( D \) is the distance from the antenna, and \( \gamma \) is the path-loss exponent (ranges from 2 to 6).

Major challenges to wireless communication systems today include (1) scarcity of spectrum, (2) ever higher data rates required, (3) multitude of environmental factors, (4) power consumption for handheld devices, (5) software complexity to support user mobility, and (6) infrastructure cost.
Jean Joseph Baptiste Fourier

March 21, 1768 to May 16, 1830
Review: Fourier Trignometric Series (for Periodic Waveforms)

Equation (2.10) should read (time $t$ is missing in the book):

$$ f(t) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right) $$

where $\omega_0 = \frac{2\pi}{T}$ and $f_0 = \frac{1}{T}$

and (Equations 2.12a, b, & c)

$$ a_0 = \frac{1}{T} \int_{0}^{T} f(t) dt \quad \text{(DC term)} $$

$$ a_n = \frac{2}{T} \int_{0}^{T} f(t) \cos(n\omega_0 t) dt \quad \text{for } n = 1, 2, 3, \text{ etc.} \quad \text{Eq. (2.12a,b,c)} $$

$$ b_n = \frac{2}{T} \int_{0}^{T} f(t) \sin(n\omega_0 t) dt \quad \text{for } n = 1, 2, 3, \text{ etc.} \quad \text{Eq. (2.12a,b,c)} $$

Agbo & Sadiku;
Section 2.5
pp. 26-27
Fourier Trigonometric Series in Amplitude-Phase Form

Equations (2.13) and (2.14) should read:

\[ f(t) = A_0 + \sum_{n=1}^{\infty} \left( A_n \cos(n\omega_0 t + \phi_n) \right) \quad \text{Eq. (2.13)} \]

Must include time \( t \)

\[ a_0 = A_0 \]

\[ A_n = \sqrt{a_n^2 + b_n^2} \quad \text{and} \]

\[ \phi_n = -\tan^{-1}\left( \frac{b_n}{a_n} \right) \]

Also known as polar form of Fourier series.

Agbo & Sadiku; Section 2.5 Page 26
Fourier Complex Exponential Series

\[ f(t) = \sum_{n = -\infty}^{\infty} C_n e^{j n \omega_0 t} \]  
Eq. (2.19)

\[ C_n = \frac{1}{T} \int_{0}^{T} f(t) \cdot e^{-j n \omega_0 t} \, dt \]

and \( C_{-n} = C_n^* \); \( C_n = |C_n| e^{j \theta_n} \)

This form comes from the Euler identity.

Agbo & Sadiku;  
Section 2.6  
pp. 33 to 39
Example: Periodic Square Wave as Sum of Sinusoids

- $f_0$
- $3f_0$
- $5f_0$
- $7f_0$

Line Spectra

Even or Odd?
Example: Periodic Square Wave (continued)

This is an odd function

\[ f(t) = \frac{4}{\pi} \left[ \sin(\pi t) + \frac{1}{3}\sin(3\pi t) + \frac{1}{5}\sin(5\pi t) + \frac{1}{7}\sin(7\pi t) + \cdots \right] \]

http://ceng.gazi.edu.tr/dsp/fourier_series/description.aspx

Question: What would make this an even function?
Analyzing Waveforms Using Fourier Series versus Waveforms from Adding Harmonics

When we analyze a waveform each harmonic is in phase with the fundamental sinusoidal component. Changing the phase of any of the harmonic sinusoidal components changes the Shape of the waveform.

In a circuit or system the internally generated harmonics of the fundamental component have different phases in general and give rise to distorted waveforms. The next several slides illustrate harmonic distortion in waveforms.
Second Harmonic Distortion in Waveform

Distortion from second harmonic being in-phase with the fundamental.

From: A. M. Niknejad, Notes from EECS 142 Lecture 7, U.C. Berkeley.
Third Harmonic Distortion in Waveform

Distortion from third harmonic being in-phase with fundamental.

From: A. M. Niknejad, Notes from EECS 142 Lecture 7, U.C. Berkeley.
Third Harmonic Distortion in Waveform (continued)

Distortion from third harmonic being out-of-phase with fundamental.

From: A. M. Niknejad, Notes from EECS 142 Lecture 7, U.C. Berkeley.
Summary of Harmonic Distortion of Fundamental

Even-Order Distortion

Odd-Order Distortion

→ DC shift

http://www.vias.org/crowhurstba/crowhurst_basic_audio_vol2_075.html
Sinusoidal Waveforms are the Building Blocks in the Fourier Series

Simple Harmonic Motion Produces a Sinusoidal Waveform

Sheet of paper unrolls in this direction
Visualizing a Signal – Time Domain & Frequency Domain

To go from time domain to frequency domain we use Fourier Transform

Source: Agilent Technologies Application Note 150, “Spectrum Analyzer Basics”
Example Where Both Sine & Cosine Terms are Required
Both even and odd parts to the waveform.

Note phase shift in the fundamental frequency sine waveform.

http://www.peterstone.name/Maplepgs/fourier.html#anchor2315207
Fourier Series versus Fourier Transform

- Fourier Series for continuous-time periodic signals → discrete spectra
- Fourier transform for continuous aperiodic signals → continuous spectra
Definition of Fourier Transform

The Fourier transform (i.e., spectrum) of \( f(t) \) is \( F(\omega) \):

\[
F(\omega) = \mathcal{F} \{ f(t) \} = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt \quad \text{Eq. (2.30)}
\]

\[
f(t) = \mathcal{F}^{-1} \{ F(\omega) \} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, d\omega \quad \text{Eq. (2.31)}
\]

Therefore, \( f(t) \iff F(\omega) \) is a Fourier Transform pair

Agbo & Sadiku;
Section 2.7;
pp. 40-41

Note: Remember \( \omega = 2\pi f \)
Fourier Transform Produces a Continuous Spectrum

\( \mathcal{F}\{f(t)\} \) gives a spectra consisting of a continuous sum of exponentials with frequencies ranging from \(-\infty\) to \(+\infty\).

\[
F(\omega) = |F(\omega)| \cdot e^{j\varphi(\omega)},
\]

where \(|F(\omega)|\) is the continuous amplitude spectrum of \(f(t)\) and \(\varphi(\omega)\) is the continuous phase spectrum of \(f(t)\).

Often only the magnitude of \(F(\omega)\) is displayed and the phase is ignored.
Example: Impulse Function $\delta(t)$

$$F(\omega) = \mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t} \, dt = e^{-j\omega t}\bigg|_{t=0} = e^{j0} = 1$$  

Eq. (2.8.1)

$\delta(t) \leftrightarrow 1$

$1 \leftrightarrow 2\pi\delta(\omega)$

Delta function has unity area.

$\delta(t) = \begin{cases} 
\infty & \text{if } t = 0 \\
0 & \text{if } t \neq 0 
\end{cases}$
Example: Fourier Transform of Single Rectangular Pulse

\[ f(t) = \text{rect}(t) = \Pi(t/\tau) = \begin{cases} A & \text{for } -\frac{\tau}{2} \leq t \leq \frac{\tau}{2} \\ 0 & \text{for all } |t| > \frac{\tau}{2} \end{cases} \]

Remember \( \omega = 2\pi f \)

\[ F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt = \int_{-\tau/2}^{\tau/2} Ae^{-j\omega t} dt \]

\[ = A \cdot \left( \frac{e^{-j\omega t}}{-j\omega} \right) \bigg|_{-\tau/2}^{\tau/2} = A\tau \cdot \left( \frac{e^{j\omega \tau/2} - e^{-j\omega \tau/2}}{2j(\omega \tau/2)} \right) \]

\[ = A\tau \cdot \frac{\sin(\omega \tau/2)}{\omega \tau/2} = A\tau \cdot \text{sinc}(\omega \tau/2) \]
Fourier Transform of Single Rectangular Pulse (continued)

\[ F(\omega) = A\tau \cdot \left[ \frac{\sin(\omega \tau/2)}{(\omega \tau/2)} \right] = A\tau \cdot \text{sinc}(\pi f \tau) \]

Note the pulse is time centered.
Properties of the Sinc Function

Definition of the sinc function:

\[ \text{sinc}(x) = \frac{\sin(x)}{x} \]

**Sinc Properties:**

1. \( \text{sinc}(x) \) is an even function of \( x \).
2. \( \text{sinc}(x) = 0 \) at points where \( \sin(x) = 0 \), that is, \( \text{sinc}(x) = 0 \) when \( x = \pm\pi, \pm2\pi, \pm3\pi, \ldots \).
3. Using *L’Hôpital’s rule*, it can be shown that \( \text{sinc}(0) = 1 \).
4. \( \text{sinc}(x) \) oscillates as \( \sin(x) \) oscillates but monotonically decreases as \( 1/x \) decreases as \( |x| \) increases.
5. \( \text{sinc}(x) \) is the Fourier transform of a single rectangular pulse.

**Warning:**

There are two definitions for \( \text{sinc}(x) \) function. They are

\[ \text{sinc}(x) = \frac{\sin(x)}{x} \quad \text{and} \quad \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \]
Periodic Pulse Train Morphing Into a Single Pulse

Frequency resolution inversely proportional to the period.

$C_k$

Fourier Transform in the limit $T \to \infty$

Figure 4.3: Take the pulse train in (a), as we increase its period, i.e., allow more time between the pulses, the fundamental frequency gets smaller, which makes the spectral lines move closer together as in (c). In the limiting case, where the period goes to $\infty$, the spectrum would become continuous.

Sinc Function Tradeoff: Pulse Duration \textit{versus} Bandwidth

1. \( G_1(f) \)
   \[
   \frac{1}{T_1} \quad \frac{-1}{T_1} \quad \frac{1}{T_1} 
   \]
   \( g_1(t) \)
   \[
   \frac{-T_1}{2} \quad \frac{T_1}{2} 
   \]

2. \( G_2(f) \)
   \[
   \frac{-1}{T_2} \quad \frac{1}{T_2} 
   \]
   \( g_2(t) / T_2 \)
   \[
   \frac{-T_2}{2} \quad \frac{T_2}{2} 
   \]

3. \( G_3(f) \)
   \[
   \frac{1}{T_3} 
   \]
   \( g_3(t) / T_3 \)
   \[
   \frac{-T_3}{2} \quad \frac{T_3}{2} 
   \]

\( T_1 > T_2 > T_3 \)

Also called the Time Scaling Property (Section 2.8.2)
Properties of Fourier Transforms

Section

2.8.1 Linearity (Superposition) Property
2.8.2 Time-Scaling Property
2.8.3 Time-Shifting Property
2.8.4 Frequency-Shifting Property
2.8.5 Time Differentiation Property
2.8.6 Frequency Differentiation Property
2.8.7 Time Integration Property
2.8.8 Time-Frequency Duality Property
2.8.9 Convolution Property

Agbo & Sadiku
Section 2.8;
pp. 46 to 58
2.8.1 Linearity (Superposition) Property

Given \( f(t) \leftrightarrow F(\omega) \) and \( g(t) \leftrightarrow G(\omega) \);

Then \( f(t) + g(t) \leftrightarrow F(\omega) + G(\omega) \)  \hspace{1cm} (additivity)

also \( kf(t) \leftrightarrow kF(\omega) \) and \( mg(t) \leftrightarrow mG(\omega) \) \hspace{1cm} (homogeneity)

Note: \( k \) and \( m \) are constants

Combining these we have,

\[
kf(t) + mg(t) \leftrightarrow kF(\omega) + mG(\omega)
\]

Hence, the Fourier Transform is a linear transformation.

This is the same definition for linearity as used in your circuits and systems course, EE 400.
2.8.2 Time Scaling Property

\[ \mathcal{F} \{ f(at) \} = \frac{1}{|a|} F \left( \frac{\omega}{a} \right) \]

Agbo & Sadiku
Section 2.8.2; pp. 46-47

\[ \mathcal{F} \{ f(at) \} = \int_{-\infty}^{\infty} f(at) e^{-j\omega t} \, dt \]

Let \( \lambda = at \) & \( d\lambda = adt \),

\[ \mathcal{F} \{ f(at) \} = \int_{-\infty}^{\infty} f(\lambda) e^{-j\omega t} \frac{d\lambda}{a} = \frac{1}{a} F \left( \frac{\omega}{a} \right) \]

Hence, \( \mathcal{F} \{ f(-t) \} = F(-\omega) = F^* (\omega) \)
Time-Scaling Property (continued)

\[
\mathcal{F}\{f(at)\} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)
\]

Time compression of a signal results in spectral expansion and time expansion of a signal results in spectral compression.
2.8.3 Time Shifting Property

\[ \mathcal{F}\{f(t - t_0)\} = e^{-j\omega t_0} F(\omega) \]

\[ \mathcal{F}\{f(t - t_0)\} = \int_{-\infty}^{ \infty} f(t - t_0)e^{-j\omega t} dt \]

Let \( \lambda = t - t_0, \ d\lambda = dt \) & \( t = \lambda + t_0 \)

\[ \mathcal{F}\{f(t - t_0)\} = \int_{-\infty}^{ \infty} f(\lambda)e^{-j\omega(\lambda+t_0)} d\lambda = \]

\[ = e^{-j\omega t_0} \int_{-\infty}^{ \infty} f(\lambda)e^{-j\omega \lambda} d\lambda = e^{-j\omega t_0} F(\omega) \]

Agbo & Sadiku
Section 2.8.3; pp. 47-48
Time-Shifting Property (continued)

\[
\mathcal{F}\{f(t - t_0)\} = e^{-j\omega t_0} F(\omega)
\]

Delaying a signal by \(t_0\) seconds does not change its amplitude spectrum, but the phase spectrum is changed by \(-2\pi ft_0\). Note that the phase spectrum shift changes linearly with frequency \(f\).

\[
|F(\omega)| = \sqrt{\left[\text{Re}(F(\omega))\right]^2 + \left[\text{Im}(F(\omega))\right]^2}
\]

A time shift produces a phase shift in its spectrum.

This time shifted pulse is both even and odd.

Both must be identical.
2.8.4 Frequency Shifting Property

\[ \mathcal{F} \{ f(t)e^{j\omega_0 t} \} = F(\omega - \omega_0) \]

\[
\mathcal{F} f(t)e^{j\omega_0 t} = \int_{-\infty}^{\infty} f(t)e^{j\omega_0 t} e^{-j\omega t} dt \\
= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t} d\lambda = F(\omega - \omega_0)
\]

Special application:

Apply to \( \cos(\omega_0 t) = \frac{1}{2} (e^{j\omega_0 t} + e^{-j\omega_0 t}) \);

\[ \mathcal{F} \{ f(t)\cos(\omega_0 t) \} = \frac{1}{2} \left( F(\omega - \omega_0) + F(\omega + \omega_0) \right) \]

Agbo & Sadiku
Section 2.8.4;
p. 48
An Important Formula to Remember in EE 442

**Euler's formula**

\[
\exp[\pm j\theta] = \cos(\theta) \pm j\sin(\theta)
\]

\[
\sin(\theta) = \frac{1}{2j} \left( \exp[j\theta] - \exp[-j\theta] \right)
\]

\[
\cos(\theta) = \frac{1}{2} \left( \exp[j\theta] + \exp[-j\theta] \right)
\]

\[
e^{\pm j(\pi/2)} = \pm j \quad \text{and} \quad e^{\pm jn\pi} = \begin{cases} 
1 & \text{for } n \text{ even} \\
-1 & \text{for } n \text{ odd} 
\end{cases}
\]

\[
a + jb = re^{j\theta} \quad \text{where} \quad r = \sqrt{a^2 + b^2} , \quad \theta = \tan^{-1}\left(\frac{b}{a}\right)
\]
Frequency Shifting Property is Very Useful in Communications

Multiplication of a signal $g(t)$ by the factor $[\cos(2\pi f_c t)]$ places $G(f)$ centered at $f = \pm f_c$.

Carrier frequency is $f_c$ & $g(t)$ is the message signal

After: B. F. Lathi & Z. Ding, 4th ed., Chapter 4, Section 4.2, Figure 4.1 (p. 181)
Frequency-Shifting Property (continued)

After: B. F. Lathi & Z. Ding, 4th ed., Chapter 3, Section 3.5, Figure 3.21 (p. 115)
Modulation Comes From Frequency Shifting Property

Given FT pair: \( f(t) \Leftrightarrow F(\omega) \)

then, \( f(t)e^{j\omega_0 t} \Leftrightarrow F(\omega - \omega_0) \)

Amplitude Modulation Example:

Audio tone:
\[ \sim \sin(\omega t) \]

Sinusoidal carrier signal:

Amplitude Modulated Signal
Fourier Transform of AM Tone Modulated Signal

\[ f(t) \]

Carrier signal \( f_c = 500 \text{ Hz} \)

Modulated AM sidebands

\[ F(\omega) \]

Only positive frequencies shown; Must include negative frequencies.

Modulation of Baseband and Carrier Signals

**Fourier Transform Properties - 8**

- Modulation
  \[ x(t) \cos 2\pi f_0 t \leftrightarrow \frac{1}{2} X(f - f_0) + \frac{1}{2} X(f + f_0) \]

  **Multiplication ↔ Convolution**

\[ x(t) \cdot \cos 2\pi f_0 t \leftrightarrow X(f) * \left[ \frac{1}{2} \delta(f - f_0) + \frac{1}{2} \delta(f + f_0) \right] \]

- \[ f_c = f_0 \]

---

**Time Domain**

- Baseband
  - Message signal

**Frequency Domain**

- Multiplication
  - \( \cos 2\pi f_0 t \leftrightarrow \frac{1}{2} \delta(f - f_0) + \frac{1}{2} \delta(f + f_0) \)

- Convolution
  - \( f_c = \text{carrier frequency} \)

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Minjoong Rim, Dongguk University

[https://slideplayer.com/slide/10939951/](https://slideplayer.com/slide/10939951/)

ES 442 Fourier Transform
Transform Duality Property

\[ g(t) \Leftrightarrow G(f) \]

and

\[ G(t) \Leftrightarrow g(-f) \]

Note the minus sign!

Because of the minus sign they are not perfectly symmetrical – See the illustration on next slide.

Agbo & Sadiku
Section 2.8.8; pp. 51-52
Illustration of Fourier Transform Duality

What does this imply?
Fourier Transform of Complex Exponentials

\[ \mathcal{F}^{-1} [ \delta(f - f_c)] = \int_{-\infty}^{\infty} \delta(f - f_c) e^{-j2\pi f t} \, df \]

Evaluate for \( f = f_c \)

\[ \mathcal{F}^{-1} [ \delta(f - f_c)] = \int_{f = f_c} \delta(f + f_c) e^{-j2\pi f_c t} \, df = e^{-j2\pi f_c t} \]

\[ \therefore \delta(f - f_c) \Leftrightarrow e^{-j2\pi f_c t} \quad \text{and} \]

\[ \mathcal{F}^{-1} [ \delta(f + f_c)] = \int_{-\infty}^{\infty} \delta(f + f_c) e^{-j2\pi f t} \, df \]

Evaluate for \( f = -f_c \)

\[ \mathcal{F}^{-1} [ \delta(f + f_c)] = \int_{f = -f_c} e^{j2\pi f_c t} \, df = e^{j2\pi f_c t} \]

\[ \therefore \delta(f + f_c) \Leftrightarrow e^{j2\pi f_c t} \]
Fourier Transform of Sinusoidal Functions

Taking \( \delta(f - f_c) \Leftrightarrow e^{-j2\pi f_c t} \) and \( \delta(f + f_c) \Leftrightarrow e^{j2\pi f_c t} \)

We use these results to find FT of \( \cos(2\pi ft) \) and \( \sin(2\pi ft) \)

Using the identities for \( \cos(2\pi ft) \) and \( \sin(2\pi ft) \),

* \( \cos(2\pi ft) = \frac{1}{2} \left[ e^{j2\pi f_c t} + e^{-j2\pi f_c t} \right] \) & \( \sin(2\pi ft) = \frac{1}{2j} \left[ e^{j2\pi f_c t} - e^{-j2\pi f_c t} \right] \)

Therefore,

\[
\begin{align*}
\cos(2\pi ft) &\Leftrightarrow \frac{1}{2} \left[ \delta(f + f_c) + \delta(f - f_c) \right], \\
\sin(2\pi ft) &\Leftrightarrow \frac{1}{2j} \left[ \delta(f + f_c) - \delta(f - f_c) \right]
\end{align*}
\]
**Summary of Several Fourier Transform Pairs**

<table>
<thead>
<tr>
<th>$g(t)$</th>
<th>$G(\omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{-at}u(t)$</td>
<td>$\frac{1}{a+j\omega}$</td>
</tr>
<tr>
<td>$e^{-at}u(-t)$</td>
<td>$\frac{1}{a-j\omega}$</td>
</tr>
<tr>
<td>$e^{j\omega t}$</td>
<td>$2a$</td>
</tr>
<tr>
<td>$te^{-at}u(t)$</td>
<td>$\frac{1}{(a+j\omega)^2}$</td>
</tr>
<tr>
<td>$\delta(t)$</td>
<td>$1$</td>
</tr>
<tr>
<td>$1$</td>
<td>$2\pi\delta(\omega)$</td>
</tr>
<tr>
<td>$e^{j\omega t}$</td>
<td>$2\pi\delta(\omega-a_0)$</td>
</tr>
<tr>
<td>$\cos \omega t$</td>
<td>$\pi[\delta(\omega-a_0) + \delta(\omega+a_0)]$</td>
</tr>
<tr>
<td>$\sin \omega t$</td>
<td>$j\pi[\delta(\omega+a_0) - \delta(\omega-a_0)]$</td>
</tr>
<tr>
<td>$u(t)$</td>
<td>$\pi\delta(\omega) + \frac{1}{j\omega}$</td>
</tr>
<tr>
<td>$\text{sgn } t$</td>
<td>$\frac{2}{j\omega}$</td>
</tr>
<tr>
<td>$\cos \omega t u(t)$</td>
<td>$\frac{\pi}{2}[\delta(\omega-a_0) + \delta(\omega+a_0)] + \frac{j\omega}{a_0^2 - \omega^2}$</td>
</tr>
<tr>
<td>$\sin \omega t u(t)$</td>
<td>$\frac{\pi}{2j}[\delta(\omega-a_0) - \delta(\omega+a_0)] + \frac{a_0}{a_0^2 - \omega^2}$</td>
</tr>
<tr>
<td>$e^{-at} \sin \omega t u(t)$</td>
<td>$\frac{a_0}{(a+j\omega)^2 + a_0^2}$</td>
</tr>
<tr>
<td>$e^{-at} \cos \omega t u(t)$</td>
<td>$\frac{a}{(a+j\omega)^2 + a_0^2}$</td>
</tr>
<tr>
<td>$\text{rect} \left(\frac{t}{T}\right)$</td>
<td>$T \text{sinc} \left(\frac{\alpha T}{2}\right)$</td>
</tr>
<tr>
<td>$\frac{W}{\pi} \text{sinc} (Wt)$</td>
<td>$\text{rect} \left(\frac{\alpha}{2W}\right)$</td>
</tr>
<tr>
<td>$\Delta \left(\frac{t}{T}\right)$</td>
<td>$\frac{T}{2} \text{sinc}^2 \left(\frac{\alpha T}{4}\right)$</td>
</tr>
<tr>
<td>$\frac{W}{2\pi} \text{sinc}^2 \left(\frac{Wt}{2}\right)$</td>
<td>$\Delta \left(\frac{\alpha}{2W}\right)$</td>
</tr>
<tr>
<td>$\sum_{n=-\infty}^{\infty} \delta(t-nT)$</td>
<td>$a_0 \sum_{n=-\infty}^{\infty} \delta(\omega-n\omega_0)$</td>
</tr>
<tr>
<td>$a_0 = \frac{2\pi}{T}$</td>
<td>$\sigma \sqrt{2\pi} e^{-\sigma^2 / 2}$</td>
</tr>
</tbody>
</table>

See Agbo & Sidiku; Table 2.5, Page 54; 
See also the Fourier Transform Pair Handout

http://media.cheggcdn.com/media/db0/db0fffe9-45f5-40a3-a05b-12a179139400/phpsu63he.png
Spectrum Analyzer Shows Frequency Domain

A **spectrum analyzer** measures the magnitude of an input signal versus frequency within the full frequency range of the instrument. It measures frequency, power, harmonics, distortion, noise, spurious signals and bandwidth.

➢ It is an electronic receiver
➢ Measure magnitude of signals
➢ Does not measure phase of signals
➢ Complements time domain

Courtesy: Keysight Technologies
Fourier Transform of Cosine Signal

\[ A \cos(2\pi f_c t) = \frac{A}{2} \left[ \delta(f + f_c) + \delta(f - f_c) \right] \]

3D View

Blue arrows indicate positive phase directions
Fourier Transform of Cosine Signal (as shown in textbooks)

\[ f_0 = \frac{1}{T_0} \]
Fourier Transform of Sine Signal

\[ B \sin(2\pi f_c t) = j \frac{B}{2} \left[ \delta(f + f_c) - \delta(f - f_c) \right] \]

We must subtract 90° from \( \cos(x) \) to get \( \sin(x) \)
Fourier Transform of Sine Signal (as usually shown in textbooks)

$B \cdot \sin(\omega_0 t)$

$\mathcal{F}$

$\delta(-f_0)$

Imaginary axis

$\mathcal{F}$

$f_0$

$-f_0$

$-j \frac{B}{2}$

$j \frac{B}{2}$
Visualizing Fourier Spectrum of Sinusoidal Signals

Note real and imaginary axes are rotated relative to prior slides.

Multiplying by $j$ is a phase shift
Fourier Transform of a Phase Shifted Sinusoidal Signal  
(with phase information as shown)

\[ R \cdot e^{j\phi} \cdot e^{j2\pi ft} + R \cdot e^{-j\phi} \cdot e^{-j2\pi ft} \]

\[ R = \sqrt{\left(\frac{A}{2}\right)^2 + \left(\frac{B}{2}\right)^2} \quad \text{and} \quad \phi = \tan^{-1}\left(-\frac{A}{B}\right) \]

Selected References


Auxiliary Slides For Introducing Sampling
Fourier Transform of Impulse Train $\delta(t)$  (Shah Function)

aka “Dirac Comb Function,” Shah Function & “Sampling Function”

Shah function ($\Pi(t)$):

$$\Pi(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{n=-\infty}^{\infty} \delta(t + nT_0)$$

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \Pi(t) \, dt = 1$$

Shah function ($\Pi(t)$):

Period $= T_0$

$$\begin{array}{c}
\cdots \\
-2T_0 \quad -T_0 \quad 0 \quad T_0 \quad 2T_0 \\
|t|
\end{array}$$

Period $= \frac{1}{T_0}$

$$\begin{array}{c}
\cdots \\
-2f_0 \quad -f_0 \quad 0 \quad f_0 \quad 2f_0 \\
f
\end{array}$$
Shah Function (Impulse Train) Applications

The sampling property is given by

\[ \Pi(t)f(t) = \sum_{n=-\infty}^{\infty} f(n)\delta(t - nT_0) \]

The “replicating property” is given by the convolution operation:

\[ \Pi(t) \star f(t) = \sum_{n=-\infty}^{\infty} f(t - nT_0) \]

Convolution theorem:

\[ g_1(t) \star g_2(t) \iff G_1(f)G_2(f) \quad \text{and} \]

\[ g_1(t)g_2(t) \iff G_1(f) \star G_2(f) \]
Sampling Function in Operation

\[ \mathcal{H}(t) f(t) = \sum_{n=-\infty}^{\infty} f(n) \delta(t - nT_0) \]
Fourier Transform of $u(t)\cdot\cos(\omega_0 t)$

Unit step $u(t)$

https://link.springer.com/chapter/10.1007/978-3-319-71437-0_9
Speech, Trumpet and Street Traffic Signals

(a) Speech time signal
(b) Music time signal
(c) Environmental time signal

https://www.mdpi.com/2076-3417/6/5/143/htm
Building the Expression for Sin(x) with Infinite Series

\[
\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} x^{2n+1}
\]