Angle Modulation (Phase & Frequency Modulation)
EE 442 Lecture 7

Spring Semester

Angle Modulation

Two forms of angle modulation

- Frequency modulation (FM)
- Phase modulation (PM)

Stamp issued 1983
FM invented 1933

Start reading Chapter 4 in Agbo & Sadiku
Modulation is the systematic alteration of a carrier wave so that it “carries” the information of the message or data signal $m(t)$.

Modulation allows for the designated frequency bands (with the carrier frequency at the center of the band) to be utilized for communication and allows for signal multiplexing.

Amplitude modulation (AM) is an analog and linear modulation process as opposed to frequency modulation (FM) and phase modulation (PM).

AM involves the variation of the carrier signal’s amplitude in direct proportion to the modulating signal $m(t)$.

AM is simple to implement and can be accomplished inexpensively with a small number of components; but AM has a low power efficiency (ratio of power in the message signal relative to the total transmitted power) and is very susceptible to noise and interference.

The landline telephone (PSTN or POTS) uses and voice signal bandwidth of 300 Hz to 3,400 Hz and a transmission voice channel of 0 to 4,000 Hz.

The Foxhole radio (from World War I) consists of an antenna, inductive coil (paired with parasitic capacitance to form a frequency selective resonator), earphones, and rectifier made from a razor blade and sharply pointed needle from a safety pin).
An amplitude modulation time-varying signal (double sideband with carrier – DSB-WC) is
\[ \phi_{AM}(t) = [A_C + m(t)] \cdot \cos(\omega_c t) \]

AM can be interpreted using phasors where the carrier of the AM signal is a phasor of constant amplitude \( A_C \) rotating CCW at frequency \( f_C \) and the modulating signal \( m(t) \) made up of a collection of slower rotating Fourier components of \( m(t) \) attached to the tip of the carrier phasor. The vector sum of the phasors gives the AM phasor.

The corresponding amplitude modulation spectrum is
\[ \Phi_{AM}(\omega) = \frac{1}{2} M(\omega - \omega_C) + \frac{1}{2} M(\omega + \omega_C) + \pi A_C \left[ \delta(\omega - \omega_C) + \delta(\omega + \omega_C) \right] \]

which is related to the frequency shift property of the Fourier transform.

The AM modulation index is defined as \( \mu = \frac{m_p}{A_C} \), where \( m_p \) is the peak amplitude of \( m(t) \). When \( \mu > 100\% \) overmodulation results in an AM waveform \( (i.e., \) envelope distortion).

The power efficiency \( \eta \) of AM is defined as
\[ \eta = \frac{\text{message power}}{\text{total power}} = \frac{P_m}{A_C^2 + P_m}, \]

where \( P_m \) is the message power. The power efficiency \( \eta \) is 11.1% when \( \mu = 0.5 \) and \( \eta \) is 33.3% when \( \mu = 1.0 \) (best case).
There are two ways to improve on the power efficiency of amplitude modulated signals: (a) suppress the carrier power (known as DSB-SC) in the transmission, and (b) eliminate both the carrier and one of the sidebands (SSB-SC).

Modulators: (a) Nonlinear component modulator, (b) switching modulator and (c) electronic multipliers (such as using a Gilbert cell).

A nonlinearity generates Taylor series terms beyond the term linear in variable v, such as $v^2$, $v^3$ and so on. Terms of $v^2$ (so-called square law behavior) and higher generate new frequencies that produce amplitude modulation.

Square-law modulators are very useful because they produce the DSB-SC AM signal that can be demonstrated from

$$\left( [A_c + m(t)] \cos(\omega_c t) \right)^2 \rightarrow m(t) \cdot \cos(\omega_c t)$$

which is the DSB-SC AM signal as desired.

The switching modulator relies upon the generation of a square-wave pulse train $p(t)$ to generate new frequencies as required to perform modulation, namely

$$p(t) = \frac{1}{2} + \frac{2}{\pi} \left[ \cos(\omega_c t) - \frac{1}{3} \cos(3\omega_c t) + \frac{1}{5} \cos(5\omega_c t) - \cdots \right]$$
The switching modulator generates DSB-SC AM signals directly.

A pn-junction is a nonlinear device in its forward biased state that makes a widely used modulator (and detector also).

An AM signal can always be demodulated using a coherent demodulator (needs a coherent carrier that exactly matches the carrier of the transmitter to recover the message signal \( m(t) \)).

However, there are two non-coherent methods to perform AM demodulation. These are (a) envelope detection and (b) rectifier detection.

Envelope demodulation depends upon performing half-wave rectification and letting the peaks of the AM waveform charge a capacitor which decays at a rate allowing for the capacitor voltage to approximately follow the envelope of the half-wave rectified waveform. The envelope recovery is proportional to message signal \( m(t) \).

For rectifier demodulation the capacitor of the envelope detector is omitted and the rectified AM signal is fed directly into a low-pass filter which recovers \( A_C + m(t) \). The DC component \( A_C \) may be removed using a series blocking capacitor.

With DSB-SC the power efficiency \( \eta \) approaches 100% because the square of amplitude \( A_C \) is zero from the elimination of the carrier.
Summary of Lecture 6 – Page 5

A mixer can be used to generate DSB-SC AM where the RF port is driven by the message signal $m(t)$ and the LO port is driven by the carrier signal $\cos(\omega_C t)$. The baseband message signal is centered about the carrier frequency in the LSB and USB even though the carrier power is zero.

For DSB-SC AM demodulation, again a mixer can be used to receive DSB-SC AM where the RF port is driven by the message signal $m(t) \cdot \cos(\omega_C t)$ and the LO port is driven by the carrier signal $\cos(\omega_C t)$. The IF port outputs $m(t) + m(t) \cdot \cos(2\omega_C t)$, thus allowing for $m(t)$ to be filtered out and recovered.

Synchronous demodulation requires detection of the carrier frequency using the DSB-SB AM signal. One way to do this is to square the incoming AM signal, filter it with a bandpass filter and divide the signal by two, thereby recovering a signal in step with the transmitted carrier frequency and use this signal to drive the LO port of the demodulating mixer.

Multipliers may be built using log and anti-log block (with op amps) to sum two inputs to give a modulated output. Another very widely used method uses the Gilbert cell which is integrated to produce a linear modulator.
Summary of Lecture 6 – Page 6

A commonly used method to generate DSB-SC AM is shown in the block diagram:

The \textbf{ring diode modulator} for DSB-SC AM is a balanced modulator that operates as a switching modulator – shown below.

You should understand how this mixer works.
MIXERS ARE USED FOR FREQUENCY CONVERSION AND FOR **HETERODYNING**. HETERODYNING USES AN ELECTRONIC CIRCUIT TO COMBINE AN INPUT RADIO FREQUENCY (RF) WITH ONE THAT IS GENERATED (LO) IN ORDER TO PRODUCE NEW FREQUENCIES: ONE THAT IS THE SUM OF THE TWO AND THE OTHER THE DIFFERENCE OF THE TWO. HETERODYNING IS TYPICALLY USED TO BAND-SHIFT INCOMING FREQUENCIES INTO INTERMEDIATE FREQUENCIES (IF) FOR DEMODULATION.

HETERODYNE RECEIVERS CAN PROVIDE (A) SELECTIVITY IN SIGNAL RECEPTION, (B) HANDLE A WIDE RANGE OF MODULATION FORMATS, AND (C) ARE CAPABLE OF ACCOMMODATING VERY HIGH FREQUENCIES (EVEN INTO THE MILLIMETER FREQUENCY BANDS).

ONE PROBLEM IN HETERODYNE RECEIVERS IS **IMAGE SIGNAL** PICKUP. MIXERS CONVERT TWO RF SIGNALS TO THE IF SIGNAL USING A SINGLE LO SIGNAL. THUS, BOTH RF SIGNALS ($\omega_{\text{RF}1}$ AND $\omega_{\text{RF}2}$) COMBINE WITH THE LO SIGNAL $\omega_{\text{LO}}$ TO GIVE TWO IF OUTPUTS [($\omega_{\text{LO}} - \omega_{\text{RF}1}$) AND ($\omega_{\text{RF}2} - \omega_{\text{LO}}$)].

THE SUPERHETERODYNE RECEIVER IS UNIVERSALLY USED IN RADIO AND A SINGLE CONVERSION STAGE SUPERHETERODYNE RECEIVER IS SHOWN BELOW.

https://en.wikipedia.org/wiki/Superheterodyne_receiver
Bandwidth efficiency can be improved with quadrature amplitude modulation (QAM). QAM involves two data streams: the I-channel and the Q-channel. Bandwidth efficiency is improved because two signals can share the same bandwidth of a channel. But this can only be done if the two modulated signals are orthogonal to each other.

Modulating one message (call it the in-phase message $m_I$) with $\cos(\omega_C t)$ and another message (call it the quadrature message $m_Q$) with $\sin(\omega_C t)$ makes the two signals orthogonal to each other. Thus, both messages can be independently modulated and demodulated.

The QAM signal is of the form, $\varphi_{QAM}(t) = m_I(t) \cdot \cos(\omega_C t) + m_Q(t) \cdot \sin(\omega_C t)$

QAM transmits two DSB-SC signals in the bandwidth of one DSB-SC signal. Interference between the two modulated signals of the same frequency is prevented by using two carriers in phase quadrature. The in-phase (I-phase) channel modulates the $\cos(\omega_C t)$ signal and the quadrature-phase (Q-phase) channel modulates the $\sin(\omega_C t)$ signal. The carriers used in the transmitter and receiver are synchronous with each other. In fact, they must be almost exactly in quadrature with each other; otherwise, they experience cochannel interference. Low-pass filters are used to extract the baseband signals $m_I(t)$ and $m_Q(t)$ in the receiver.

QAM is used extensively as a modulation scheme for digital telecommunication systems, such as in 802.11 Wi-Fi standards.
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QAM transmitter and receiver block diagram:

Effect of error $\Delta \omega$ in carrier frequencies between the in-phase and the quadrature channels.

$$y_1(t) = m_1(t)\cos(\Delta \omega t) - m_Q(t)\sin(\Delta \omega t)$$

$$y_Q(t) = m_Q(t)\cos(\Delta \omega t) + m_1(t)\sin(\Delta \omega t)$$
Single-sideband AM (SSB AM) is the most efficient AM signal of any AM transmission format with respect to efficient use of bandwidth (100% efficient).

The phase-sift method of generating of AM SSB.

\[
\varphi_{SSB}(t) = m(t) \cos(\omega_c t) \pm m_h(t) \sin(\omega_c t)
\]

where minus sign applies to USB and plus sign applies to the LSB.

\( m_h(t) \) is \( m(t) \) phase delayed by \(-\pi/2\)
The phase shift function labelled **Hilbert Transform** performs the following phase shift function: Given a signal, for positive frequencies, multiply it by $-j$ (phase shift by -90 deg) and for negative frequencies, multiply by $+j$ (or +90 deg).

The phase shift method uses two balanced (and identical) to eliminate the carrier. Then the phase shift is used to cancel one of the sidebands (it can be either the upper sideband or the lower sideband).

With digital signals the closest digital modulation format is **pulse amplitude modulation (PAM)**.
Angle Modulation

Angle modulation viewed as FM or PM

https://semesters.in/tag/equation-for-pm-wave/
Some Applications for Various Modulation Techniques

<table>
<thead>
<tr>
<th>Application</th>
<th>Type of Modulation</th>
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<tbody>
<tr>
<td>AM broadcast radio</td>
<td>AM</td>
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<tr>
<td>FM broadcast radio</td>
<td>FM</td>
</tr>
<tr>
<td>FM stereo multiplex sound</td>
<td>DSB (AM) and FM</td>
</tr>
<tr>
<td>TV sound</td>
<td>FM</td>
</tr>
<tr>
<td>TV picture (video)</td>
<td>AM, VSB</td>
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<tr>
<td>TV color signals</td>
<td>Quadrature DSB (AM)</td>
</tr>
<tr>
<td>Cellular telephone</td>
<td>FM, FSK, PSK</td>
</tr>
<tr>
<td>Cordless telephone</td>
<td>FM, PSK</td>
</tr>
<tr>
<td>Fax machine</td>
<td>FM, QAM (AM plus PSK)</td>
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<tr>
<td>Aircraft radio</td>
<td>AM</td>
</tr>
<tr>
<td>Marine radio</td>
<td>FM and SSB (AM)</td>
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<td>Mobile and handheld radio</td>
<td>FM</td>
</tr>
<tr>
<td>Citizens band radio</td>
<td>AM and SSB (AM)</td>
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<tr>
<td>Amateur radio</td>
<td>FM and SSB (AM)</td>
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<tr>
<td>Computer modems</td>
<td>FSK, PSK, QAM (AM plus PSK)</td>
</tr>
<tr>
<td>Garage door opener</td>
<td>OOK</td>
</tr>
<tr>
<td>TV remote control</td>
<td>OOK</td>
</tr>
<tr>
<td>VCR</td>
<td>FM</td>
</tr>
<tr>
<td>Family Radio service</td>
<td>FM</td>
</tr>
</tbody>
</table>

Not a complete list of applications.

We have studied AM, next is FM and PM.
Amplitude, Frequency and Phase Modulation

With few exceptions, Phase Modulation (PM) is used predominantly in digital communication.

Remember that \( f = \frac{d\phi}{dt} \)
Illustrating AM, PM and FM Signals

Carrier Wave

Modulating Signal $m(t)$

time

Carrier signal

AM Modulated Signal

PM Modulated Signal

FM Modulated Signal

Reference: Lathi & Ding

100% modulation shown

Angle Modulation

$AM \sim \frac{dm(t)}{dt}$

$PM \sim \frac{dm(t)}{dt}$

$FM \sim m(t)$

$dt$
Focus Upon an FM Signal Modulated by a Single-Tone

The diagram illustrates a single-tone modulating signal $m(t)$ and its effect on the FM signal. At the maximum signal frequency increase, the signal frequency changes. At no signal frequency change (center frequency), there is no frequency change. At the maximum signal frequency decrease, the signal frequency decreases. The diagram shows the relationship between the baseband signal and the modulated FM signal.
Comparing AM, PM and FM for a Ramp $m(t)$

- Carrier: $\cos(\omega_c t)$
- Message: $m(t)$
- Amplitude modulation
- Phase modulation: $\frac{dm(t)}{dt}$
- Frequency modulation: $m(t)$

[Link to the lecture](https://www.princeton.edu/~mvaezi/ece3770/ECE3770_Lecture7.pdf)
General Observations on FM and PM Waveforms

1. Both FM and PM waveforms are identical except for a time shift, when $m(t)$ is a sinusoidal signal.

2. For FM, the maximum frequency deviation occurs when modulating signal is at its peak values (i.e., at $+m_p$ and $-m_p$).

3. For PM, the maximum frequency deviation takes place at the zero crossings of the modulating signal $m(t)$.

4. It is generally difficult to know from looking at a waveform whether the modulation is FM or PM.

5. The message resides in the zero-crossings alone, provided the carrier frequency is large compared to frequency content of $m(t)$.

6. The modulated waveform doesn’t resemble the message waveform.

Reference: Carlson & Crilly, 5th ed., Section 5.1, pages 208 to 212.
Advantages of Angle Modulation

1. Angle modulation is **resistant to propagation-induced selective fading** because the amplitude variations don’t contain information.

2. Angle modulation is very efficient in **rejecting interference** (i.e., it minimizes the effect of amplitude noise on the signal transmission).

3. Angle modulation allows for more **efficient use of transmitter power**.

4. Angle modulation can **handle a greater dynamic range** in the modulating signal without distortion (as would occur in AM).

5. Wideband FM gives significant **improvement in the signal-to-noise ratio** at the output and is proportional to the square of the modulation index $\beta$, where

$$\beta = \frac{\Delta f}{B}$$

\[
\begin{align*}
B & \quad \text{Bandwidth} \\
\Delta f & \quad \text{Frequency deviation}
\end{align*}
\]
Phase-Frequency Relationship When Frequency is Constant

\[ \varphi(t) = A_C \cos(\theta(t)) \]

\( \theta(t) \) is generalized angle

\[ \varphi(t) = A_C \cos(\omega_C t + \theta_0) \]

No modulation

\[ \omega_C t + \theta_0 \]

\( \theta_0 \) is constant

Slope: \[ \omega_i(t) = \frac{d\theta(t)}{dt} \bigg|_{t=t_i} = \omega_C \]

\( \theta_0 \)

\( t \)

No modulation
Concept of Instantaneous Frequency

\[ \varphi(t) = A_C \cos(\theta(t)) \]

\[ \theta(t) \text{ is generalized angle} \]

\[ \varphi(t) = A_C \cos(\omega_C t + \theta_0) \]

\[ \omega_C \text{ is constant} \]

Slope: \[ \omega_i(t) = \left. \frac{d\theta(t)}{dt} \right|_{t=t_i} > \omega_C \]
Angle Modulation Gives PM and FM

\[ \omega_i(t) = \frac{d\theta(t)}{dt} \bigg|_{t=t_i} \quad \text{and} \quad \theta(t) = \int_{-\infty}^{t} \omega_i(\lambda) \, d\lambda \]

Frequency modulation and phase modulation are closely related!
## Comparing Frequency Modulation to Phase Modulation

<table>
<thead>
<tr>
<th>No.</th>
<th>Frequency Modulation (FM)</th>
<th>Phase Modulation (PM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Frequency deviation is proportional to modulating signal $m(t)$</td>
<td>Phase deviation is proportional to modulating signal $m(t)$</td>
</tr>
<tr>
<td>2</td>
<td>Noise immunity is superior to PM (and of course AM)</td>
<td>Noise immunity better than AM, but not FM</td>
</tr>
<tr>
<td>3</td>
<td>Signal-to-noise ratio (SNR) is better than PM (and of course AM)</td>
<td>Signal-to-noise ratio (SNR) is not quite as good as with FM</td>
</tr>
<tr>
<td>4</td>
<td>FM is widely used for commercial broadcast radio (88 MHz to 108 MHz)</td>
<td>PM is primarily used for mobile radio services</td>
</tr>
<tr>
<td>5</td>
<td>Modulation index is proportional to modulating signal $m(t)$ as well as the modulating frequency $f_m$</td>
<td>Modulation index is proportional to modulating signal $m(t)$</td>
</tr>
</tbody>
</table>
FM has better noise (or RFI) rejection than AM, as shown in this dramatic New York publicity demonstration by General Electric in 1940. The radio contained both AM and FM receivers. With a million-volt arc as a source of interference behind it, the AM receiver produced only a roar of static, while the FM receiver clearly reproduced a music program from Armstrong's experimental FM transmitter W2XMSN in New Jersey.

https://en.wikipedia.org/wiki/Frequency_modulation

Note: RFI stands for radio frequency interference.
Phase Modulation (PM)

\[ \theta_i(t) = \omega_C t + \theta_0 + k_p m(t) ; \quad \text{Usually we set } \theta_0 = 0 , \]

\[ \varphi_{PM}(t) = A_C \cos(\omega_C t + k_p m(t)) \]

The instantaneous angular frequency (in radians/second) is

\[ \omega_i(t) = \frac{d\theta_i(t)}{dt} = \omega_C + k_p \frac{dm(t)}{dt} = \omega_C + k_p m'(t) \]

In phase modulation (PM) the instantaneous angular frequency \( \omega_i \) varies linearly with the time derivative of the message signal \( m(t) \) [denoted here by \( m'(t) \)].

\( k_p \) is the phase-deviation (sensitivity) constant. Units: radians/volt
[Actually it is radians/unit of the parameter \( m(t) \).]
Frequency Modulation (FM)

But in frequency modulation the instantaneous angular frequency \( \omega_i \) varies linearly with the modulating signal \( m(t) \),

\[
\omega_i(t) = \omega_C + k_f m(t)
\]

\[
\theta_i(t) = \int_{-\infty}^{t} \left[ \omega_C + k_f m(\lambda) \right] d\lambda = \omega_C t + k_f \int_{-\infty}^{t} m(\lambda) d\lambda
\]

\( k_f \) is frequency-deviation (sensitivity) constant. Units: radians/volt-sec.

Then

\[
\varphi_{FM}(t) = A_C \cos \left( \omega_C t + k_f \int_{-\infty}^{t} m(\lambda) d\lambda \right)
\]

FM and PM are related to each other.
In PM the angle is directly proportional to \( m(t) \).
In FM the angle is directly proportional to the integral \( \int m(t) dt \).

Agbo & Sadiku
Section 4.2; p. 159
## Summary

Message signal is \( m(t) \)

**Definition:** Instantaneous frequency is \( \omega_i(t) = \frac{d\theta_i(t)}{dt} \)

<table>
<thead>
<tr>
<th></th>
<th>Phase Modulation</th>
<th>Frequency Modulation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Angle</strong></td>
<td>( \theta_i(t) = \omega_C t + k_p m(t) )</td>
<td>( \theta_i(t) = \omega_C t + k_f \int_{-\infty}^{t} m(\lambda) d\lambda )</td>
</tr>
<tr>
<td><strong>Frequency</strong></td>
<td>( \omega_i = \omega_C + k_p \frac{dm(t)}{dt} )</td>
<td>( \omega_i = \omega_C + k_f m(t) )</td>
</tr>
</tbody>
</table>

In phase modulation \( m(t) \) drives the time variation of phase \( \theta_i \).

In frequency modulation \( m(t) \) drives the time variation of frequency \( f_c \).
We require that $H(j\omega)$ be a reversible (or invertible) operation so that $m(t)$ is recoverable.
Both FM and PM Generation are Nonlinear Processes

Consider a phase modulated signal:

Let \( s(t) = A_C \cos \left( \omega_C t + k_p [m_1(t) + m_2(t)] \right) \)

If \( s_1(t) = A_C \cos \left( \omega_C t + k_p m_1(t) \right) \), and
\( s_2(t) = A_C \cos \left( \omega_C t + k_p m_2(t) \right) \)

It then holds that
\[ s_1(t) + s_2(t) \neq s(t) \quad \therefore \text{additivity fails} \]

So PM can't be linear.

The same argument holds for FM.

Note: Linearity requires both additivity and homogeneity to hold.
Modulation Index $\beta$ for Angle Modulation

Let the peak values of the message signal $m(t)$ and its first derivative $m'(t)$ be represented by

- Peak value of $m(t) = m_p = \frac{1}{2}(m_{\text{max}} - m_{\text{min}})$
- Peak value of $m'(t) [= dm(t)/dt] = m'_p$

**Frequency Deviation** is the maximum deviation of the instantaneous modulated carrier frequency relative to the unmodulated carrier frequency. It is (symbolically) represented by either $\Delta \omega$ or $\Delta f$.

- **FM:** $\Delta \omega = k_f m_p$ or $\Delta f = \frac{k_f m_p}{2\pi}$
- **PM:** $\Delta \omega = k_p m'_p$ or $\Delta f = \frac{k_p m'_p}{2\pi}$

The ratio of the frequency deviation $\Delta f$ to the message signal’s bandwidth $B$ Is called the Frequency Deviation Ratio or the **Modulation Index**, and is denoted by $\beta$ (unitless).

$$\beta = \frac{\Delta f}{B} = \frac{\Delta \omega}{2\pi B}$$
Equations for FM Wave with Single-Tone Modulation

Carrier signal \( A_C \cos(\omega_C t) \) (volts)
Carrier frequency \( \omega_C = 2\pi f_C \) (radians/sec)
Modulating wave \( m(t) \) \( A_m \cos(\omega_m t) \) \textit{Single-tone modulation}
Modulating frequency \( \omega_m = 2\pi f_m \) (radians/sec)
Deviation sensitivity \( k_f \) (radians/volt-second)
Frequency deviation \( \Delta \omega = k_f A_m \) (radians/sec)

Modulation Index \( \beta = \frac{\Delta f}{f_m} = \frac{\Delta \omega}{\omega_m} = \frac{k_f A_m}{\omega_m} \) (unitless)

Instantaneous frequency \( f_i = f_C + k_f A_m \frac{\cos(\omega_m t)}{2\pi} = f_C + \Delta f \cos(\omega_m t) \)

Remember \( \varphi_{FM}(t) = A_C \left[ \cos \left( \omega_C t + k_f \int_{-\infty}^{t} m(\lambda) d\lambda \right) \right] \), generally

Tone modulated wave \( \varphi_{FM}(t) = A_C \left[ \cos \left( \omega_C t + \frac{k_f A_m}{\omega_m} \sin(\omega_m t) \right) \right] \)
or \( \varphi_{FM}(t) = A_C \left[ \cos(\omega_C t + \beta \sin(\omega_m t)) \right] \)
### Summary of Mathematical Equations for FM and PM

<table>
<thead>
<tr>
<th>Type of Modulation</th>
<th>Modulating Signal</th>
<th>Angle Modulated Wave</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase modulation</td>
<td>$m(t)$</td>
<td>$A_C \cdot \cos \left( \omega_C t + k_p m(t) \right)$</td>
</tr>
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</tr>
<tr>
<td>Phase modulation</td>
<td><strong>Tone:</strong> $m(t) = A_m \cdot \cos(\omega_m t)$</td>
<td>$A_C \cdot \cos \left( \omega_C t + k_p A_m \cos(\omega_C t) \right)$</td>
</tr>
<tr>
<td>Frequency modulation</td>
<td><strong>Tone:</strong> $m(t) = A_m \cdot \cos(\omega_m t)$</td>
<td>$A_C \cdot \cos \left( \omega_C t + \frac{k_f A_m}{\omega_m} \sin(\omega_C t) \right)$</td>
</tr>
</tbody>
</table>

\[ \beta \triangleq \frac{k_f A_m}{\omega_m} \]
Example

- A single-tone FM signal is

\[ \varphi_{FM}(t) = 10\left[ \cos\left(2\pi(10^6)t + 8\sin(2\pi(10^3)t)\right) \right] \]

Determine
a) the carrier frequency \( f_c \)
b) the modulation index \( \beta \)
c) the peak frequency deviation \( \Delta f \)
Solution to Example

Start with the basic FM equation:

\[ \phi_{FM}(t) = A_C \left[ \cos(2\pi f_C t + \beta \sin(2\pi f_m t)) \right] \]

Compare this to

\[ \phi_{FM}(t) = 10 \left[ \cos(2\pi (10^6) t + 8 \sin(2\pi (10^3) t)) \right] \]

(a) We see that \( f_C = 1,000,000 \) Hz & \( f_m = 1000 \) Hz.
(b) The modulation index is \( \beta = 8 \).
(c) The peak deviation frequency \( \Delta f \) is

\[ \Delta f = \beta \cdot f_m = 8 \cdot 1000 = 8,000 \text{ Hz} \]

Note: \( \Delta f / f_C \) is 0.008 or 0.8 % deviation frequency to carrier frequency.
Average Power of a FM or PM Wave

The amplitude $A_C$ is constant in a phase modulated or a frequency modulated signal. RF power does not depend upon the frequency or the phase of the waveform.

$$\varphi_{FM\ or\ PM}(t) = A_C \cos\left[ \omega_C t + g(k_k, m(t)) \right]$$

Average Power $= \frac{A_C^2}{2}$ (always)

This is a result of FM and PM signals being constant amplitude.

Note: $k_k$ becomes $k_f$ for FM and $k_p$ for PM.
Average Power of a FM or PM Wave

Problem:

Consider an angle modulated signal given by

\[ \phi(t) = 6 \cdot \left[ \cos \left( 2\pi \times 10^6 t + 2 \cdot \sin(8000\pi t) \right) \right] \text{ volts} \]

What is the average power of this signal?

Solution:

Average power \( P_c = \frac{A_c^2}{2} \) where \( A_c = 6 \text{ volts} \)

Therefore, \( P_c = \frac{6^2}{2} = \frac{36}{2} = 18 \text{ watts} \) (assumes 1 ohm resistance)

Note that the result does not depend upon it being FM or PM.
Comparison of FM (or PM) to AM

<table>
<thead>
<tr>
<th>#</th>
<th>Frequency Modulation (FM)</th>
<th>Amplitude Modulation (AM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>FM receivers have better noise immunity</td>
<td>AM receivers are very susceptible to noise</td>
</tr>
<tr>
<td>2</td>
<td>Noise immunity can be improved by increasing the frequency deviation</td>
<td>The only option in AM is to increase the transmission power</td>
</tr>
<tr>
<td>3</td>
<td>Bandwidth requirement is greater and depends upon modulation index</td>
<td>AM bandwidth is less than FM or PM and doesn’t depend upon a modulation index</td>
</tr>
<tr>
<td>4</td>
<td>FM (or PM) transmitters and receivers are more complex than for AM</td>
<td>AM transmitters and receivers are less complex than for FM (or PM)</td>
</tr>
<tr>
<td>5</td>
<td>All transmitted power is useful so FM is very efficient</td>
<td>Power is wasted in transmitting the carrier and double sidebands in DSB (but DSB-SC &amp; SSB addresses this)</td>
</tr>
</tbody>
</table>
AM, FM and PM Waveforms for Single-Tone $m(t)$

- **Carrier Wave**
- **Modulating Signal $m(t)$**

- **AM Modulated Signal**
- **PM Modulated Signal**
- **FM Modulated Signal**

100% modulation shown

Focus upon frequency

Angle Modulation

Reference: Lathi & Ding

$$\text{Reference: Lathi & Ding}$$

$$dm(t)$$

$$dt$$

$$m(t)$$

$$t$$
FM and PM Examples

Sketch FM and PM waveforms for the modulating signal $m(t)$. The constants $k_f$ and $k_p$ are $2\pi \times 10^5$ and $10\pi$, respectively. Carrier frequency $f_c = 100$ MHz.

$$f_i = f_c + \frac{k_f}{2\pi} m(t) = 1 \times 10^8 + 1 \times 10^5 \cdot m(t);$$

$\left(f_i\right)_{\text{min}} = -1$ and $\left(f_i\right)_{\text{max}} = 1$

$\left(f_i\right)'_{\text{min}} = 10^8 + 10^5(-1) = 99.9$ MHz,

$\left(f_i\right)'_{\text{max}} = 10^8 + 10^5(1) = 100.1$ MHz

$$f_i = f_c + \frac{k_p}{2\pi} m'(t) = 1 \times 10^8 + 5 \cdot m'(t);$$

$\left(m'\right)_{\text{min}} = -20,000$ and $\left(m'\right)_{\text{max}} = 20,000$

$\left(f_i\right)'_{\text{min}} = 10^8 + 5(-20,000) = 99.9$ MHz,

$\left(f_i\right)'_{\text{max}} = 10^8 + 5(+20,000) = 100.1$ MHz

Fig. 5.4; p. 256 of 4th ed., Lathi & Ding
Digital Frequency Shift Keying is Related to FM

Sketch the FM waveform for the modulating signal $m(t)$. The constant $k_f$ is $2\pi \times 10^5$. Carrier frequency $f_c = 100 \text{ MHz}$.

Since $m(t)$ switches from +1 to -1 and vice versa, the FM wave frequency switches between 99.9 MHz and 100.1 MHz. This is called **Frequency Shift Keying (FSK)** and is a digital communication format.

$$f_i = f_c + \frac{k_f}{2\pi} m(t) = 1\times 10^8 + 1\times 10^5 m(t)$$

**Fig. 5.5; p. 258 of 4th ed., Lathi & Ding**
Example – continued

Sketch the PM waveform for the modulating signal $m(t)$ from prior slide. The constant $k_p$ equals $\pi/2$. Carrier frequency $f_c = 100$ MHz.

\[ f_i = f_c + \frac{k_p}{2\pi} \frac{dm(t)}{dt} = 1 \times 10^8 + \frac{1}{4} \frac{dm(t)}{dt} \]

This is carrier PM by a digital signal – it is **Phase Shift Keying (PSK)** because the digital data is represented by phase of the carrier wave.

Evaluate the instantaneous jumps by considering:

\[ \varphi_{PM}(t) = A_C \cos \left[ \omega_C t + k_p m_d(t) \right] = A_C \cos \left[ \omega_C t + \frac{\pi}{2} m_d(t) \right] \]

\[ k_p m_d(t) \sim (-\pi, \pi) \]

\[ \varphi_{PM}(t) = A_C \sin(\omega_C t) \quad \text{when} \quad m(t) = -1 \]

\[ \varphi_{PM}(t) = -A_C \sin(\omega_C t) \quad \text{when} \quad m(t) = 1 \]

where jump in $m_d(t) = (1) - (-1) = 2$ or $(-1) - (1) = -2$

**Fig. 5.5; p. 258 of 4th ed., Lathi & Ding**
Generalized Angle Modulation

Agbo & Sadiku; Section 4.2 & 4.3 on pages 158 to 168

Start with equation (4.8) on page 159, which is

\[ \phi_A(t) = A_C \cdot \cos[\omega_C t + k \cdot \gamma(t)] \text{ where } \gamma(t) = m(t) * h(t) = \int_{-\infty}^{t} m(\lambda)h(t - \lambda)d\lambda \]

with \( h(t) = \delta(t) \) for PM; \( h(t) = u(t) \) for FM

Suppose we use the exponential carrier \( A_C e^{j\omega_C t} \) instead of \( A_C \cos(\omega_C t) \),
then the form for generalized angle modulation becomes

\[ \phi_A(t) = A_C \cdot e^{j(\omega_C t + k \gamma(t))} = A_C \cdot e^{j(\omega_C t)} \cdot e^{jk\gamma(t)} \]

where \( k \to k_p \) for PM; \( k \to k_f \) for FM

\[ \phi_{PM}(t) = \text{Re}[\phi_A(t)] = \text{Re}[A_C \cdot e^{j\omega_C t} \cdot e^{jk_p\gamma(t)}]; \text{ where } \gamma(t) = m(t) * \delta(t) = m(t) \]

and

\[ \phi_{FM}(t) = \text{Re}[\phi_A(t)] = \text{Re}[A_C \cdot e^{j\omega_C t} \cdot e^{jk_f\gamma(t)}]; \text{ where } \gamma(t) = \int_{-\infty}^{t} m(\lambda)d\lambda \]
Generalized Angle Modulation (continued)

Agbo & Sadiku; Section 4.2 & 4.3 on pages 158 to 168

Consider first Frequency Modulation (FM),

$$\phi_{FM}(t) = \text{Re}\left[\phi_A(t)\right] = \text{Re}\left[A_C \cdot e^{j\omega_c t} \sum_{n=0}^{\infty} \frac{j^n k_f^\gamma^n(t)}{n!}\right]$$

$$\phi_{FM}(t) = \text{Re}\left[A_C \cdot e^{j\omega_c t} \left(1 + jk_f \gamma(t) - \frac{k_f^2 \gamma^2(t)}{2!} + \frac{jk_f^3 \gamma^3(t)}{3!} - \cdots\right)\right]$$

Now take the real part of the expression above,

$$\phi_{FM}(t) = A_C \left[\cos(\omega_c t) - k_f \gamma(t) \sin(\omega_c t) - \frac{k_f^2 \gamma^2(t)}{2!} \cos(\omega_c t) + \cdots\right]$$

Note: $m(t)$ has a bandwidth = $B$ Hz and $\gamma(t)$ has a bandwidth = $B$ Hz, but $\gamma^n(t)$ has a bandwidth = $nB$ Hz; as $n \to \infty$, bandwidth $\to \infty$

Conclusion: The instantaneous frequency deviations are symmetrical about carrier frequency $\omega_c$, thus, FM is double side-banded. The effective FM bandwidth = $2nB$ Hz.
Generalized Angle Modulation (continued)

Agbo & Sadiku; Section 4.2 & 4.3 on pages 158 to 168

Consider the case where \( k_f \) is small, meaning that \(|k_f \gamma(t)| << 1\).
It is commonly referred to as **narrowband FM** (NBFM). We take only the first two terms in the expansion for \( \phi_{FM}(t) \).

\[
\phi_{FM}(t) \approx A_c \left[ \cos(\omega_c t) - k_f \gamma(t) \sin(\omega_c t) \right]
\]

By analogy, we can apply the same analysis for Phase Modulation (PM).
For PM, if \( k_p \) is small, then \(|k_p \gamma(t)| << 1\). This is known as **narrowband PM** (NBPM).

\[
\phi_{PM}(t) \approx A_c \cdot \cos(\omega_c t) - A_c k_p \left[ \int_{-\infty}^{t} m(\lambda) d\lambda \right] \sin(\omega_c t)
\]

Using these results allows us to generate narrowband FM and PM with the block diagrams on the next slide (slide #46):
Generation of Narrowband FM and PM

\[ m(t) \xrightarrow{\int} -A_c \cdot \sin(\omega_c t) \xrightarrow{\pi/2} A_c \cdot \cos(\omega_c t) \]

\[ + \]

\[ m(t) \xrightarrow{k_f} + \]

\[ NBFM \]

\[ m(t) \xrightarrow{k_p} -A_c \cdot \sin(\omega_c t) \xrightarrow{\pi/2} A_c \cdot \cos(\omega_c t) \]

\[ + \]

\[ NBPM \]

Agbo & Sadiku; Figure 4.5 on page 168
Modulation Index $\beta$ Parameter in Angle Modulation

Parameter $\beta$ is the modulation index for angle modulation.

$\beta$ is used to differentiate between narrowband angle modulation and wideband angle modulation.

- Narrowband angle modulation requires $\beta << 1$ (Typically < 0.3)
- Wideband angle modulation requires $\beta >> 1$ (Typically > 5.0)

Equivalently,

- Narrowband angle modulation requires $\Delta f << B$
- Wideband angle modulation requires $\Delta f >> B$

Comments:
1. Narrowband FM has about the same bandwidth as that of AM.
2. Commercial (broadcast) FM is wideband FM (required due to its superior noise performance).
3. Why even consider narrowband FM? Two reasons:
   a. NBFM is easier to generate than WBFM.
   b. It is commonly used as the first step in generating WBFM.
Narrowband FM with Tone Modulation

Let \( m(t) = A_m \cos(\omega_m t) \), then \( m_p = A_m \); \( \omega_m = 2\pi B \); and \( \beta = \frac{k_f m_p}{\omega_m} = \frac{k_f A_m}{\omega_m} \)

Then \( k_f \int_{-\infty}^{t} m(\lambda) d\lambda = k_f \int_{-\infty}^{t} A_m \cos(\omega_m \lambda) d\lambda = \frac{k_f A_m}{\omega_m} \cdot \sin(\omega_m t) \)

The time-domain NBFM signal is
\[
\phi_{FM}(t) \cong A_C \cos(\omega_c t) - \beta A_C \sin(\omega_m t) \cdot \sin(\omega_c t) \quad \text{Equation (4.18)}
\]
\[
\phi_{FM}(t) \cong A_C \cos(\omega_c t) + \frac{1}{2} \beta A_C \cos((\omega_c + \omega_m) t) - \frac{1}{2} \beta A_C \cos((\omega_c - \omega_m) t) \quad \text{Equation (4.19)}
\]

In comparing to AM:

The 2\(^{nd}\) term is the upper sideband and the 3\(^{rd}\) term is the lower sideband.
Narrowband FM (NBFM)

\[ \phi_{FM}(t) \approx A_C \cos(\omega_c t) + \frac{1}{2} \beta A_C \cos((\omega_c + \omega_m) t) - \frac{1}{2} \beta A_C \cos((\omega_c - \omega_m) t) \]

Tone modulation \( \sim \cos(\omega_m t) \)

Sidebands are in quadrature.

NBPM requires \( \beta \ll 1 \) radian
(generally less than 0.3 radian)
Narrowband FM (NBFM)

\[ \omega_C \text{ rotates faster than } \omega_m \]

Phasor lengths adjust to keep constant \( A_C \).
Review: Phasor Interpretation of AM DSB with Carrier

$\omega_C$ rotates faster than $\omega_m$

$\omega_m = |\omega_{us}| = |\omega_{ls}|$

Spectrum:

$\omega_C - \omega_m$  lower sideband

$\omega_C$  upper sideband

$\omega_C + \omega_m$  upper sideband

DSB  AM
Narrowband FM Example (Example 4.4)

Exercise: The message signal input to a modulator is $m(t) = 4 \cdot \cos(2\pi \times 10^4 t)$ and the carrier is $10 \cdot \cos(\pi \times 10^8 t)$. If frequency modulation is performed with $k_f = 1000\pi$, verify that the modulated signal meets the criteria of being narrowband FM. Also, obtain an expression for its spectrum and sketch this spectrum.

Solution: From: Agbo & Sadiku; page 170

First we calculate the modulation index $\beta$

$$\beta = k_f \frac{A_m}{\omega_m} = 1000\pi \left( \frac{4}{2\pi \times 10^4} \right) = 0.2; \quad \beta < 0.3 \Rightarrow NBFM$$

$$A_c = 10 \quad \text{thus,} \quad \frac{1}{2}\beta A_c = \frac{1}{2}(0.2)(10) = 1$$

Using the equation from slide #49:

$$\phi_{FM}(t) \cong A_c \cos(\omega_c t) + \frac{1}{2} \beta A_c \cos((\omega_c + \omega_m) t) - \frac{1}{2} \beta A_c \cos((\omega_c - \omega_m) t)$$

$$\phi_{FM}(t) \cong 10 \cdot \cos(\omega_c t) + \cos((\omega_c + \omega_m) t) - \cos((\omega_c - \omega_m) t)$$
Narrowband FM Example (Example 4.4 continued)

\[ \phi_{FM}(t) \cong 10 \cdot \cos(\omega_c t) + \cos((\omega_c + \omega_m)t) - \cos((\omega_c - \omega_m)t) \]

The corresponding expression for the spectrum becomes

\[ \Phi_{FM}(\omega) = 10\pi \left[ \delta(\omega + \omega_c) + \delta(\omega - \omega_c) \right] + \pi \left[ \delta(\omega + \omega_c + \omega_m) + \delta(\omega - \omega_c - \omega_m) \right] - \pi \left[ \delta(\omega + \omega_c - \omega_m) + \delta(\omega - \omega_c + \omega_m) \right] \]

where \( \omega_c = 10^8 \pi \) radians/sec and \( \omega_m = 2\pi \times 10^4 \) radians/sec

\[ \Phi_{FM}(\omega) \]

\[ 10\pi \]

\[ \pi \]

\[ -\omega_c \quad -\pi \quad \omega_c \]

\[ \omega \]

\[ (-\omega_c - \omega_m) \quad (-\omega_c + \omega_m) \quad (\omega_c - \omega_m) \quad (\omega_c + \omega_m) \]

Bandwidth = \( 2\omega_m \)
Wideband FM (WBFM)

WBFM requires $\beta >> 1$ radian (much more complicated)

For wideband FM we have a nonlinear process, with single-tone modulation:

$$\phi_{FM}^{WB}(t) = \text{Re} \left[ A_C \exp \left( j\omega_C t + j\beta \sin(\omega_m t) \right) \right]$$

We need to expand the exponential in a Fourier series in order to analyze $\phi_{FM}^{WB}(t)$. The solution has an expansion in Bessel functions:

$$\phi_{FM}^{WB}(t) = A_C \sum_{n=-\infty}^{\infty} J_n(\beta) \cdot \cos \left( 2\pi(f_c + nf_m)t \right)$$

where the coefficients $J_n(\beta)$ are Bessel functions.


We will not cover this section in EE 442 but rather focus upon the physical interpretation of FM spectrum spread.

Modulation Index

$$\beta = \frac{\Delta f}{f_m} = \frac{\Delta f}{B}$$
Digression: Bessel Functions (of the 1st kind)

Bessel functions have many applications: cylindrical waveguides, vibrational modes on circular membrane, FM modulation synthesis, acoustic vibrations, etc.

\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0 \]

http://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html

https://www.cefns.nau.edu/~schulz/Bessel/J02.html
WBFM (or WBPM) Requires More bandwidth Than AM

- Carrier Signal (frequency $f_C$)
- Message Signal (frequency $f_m$)
- Amplitude Modulated Signal
- Frequency Modulated (FM) Signal
**Single-Tone FM Spectra as Function of Modulation Index $\beta$**

$f_m$ constant

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>Number of Sidebands$^\dagger$</th>
<th>Bandwidth</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>2</td>
<td>$2f_m$</td>
</tr>
<tr>
<td>0.3</td>
<td>4</td>
<td>$4f_m$</td>
</tr>
<tr>
<td>0.5</td>
<td>4</td>
<td>$4f_m$</td>
</tr>
<tr>
<td>1.0</td>
<td>6</td>
<td>$6f_m$</td>
</tr>
<tr>
<td>2.0</td>
<td>8</td>
<td>$8f_m$</td>
</tr>
<tr>
<td>5.0</td>
<td>16</td>
<td>$16f_m$</td>
</tr>
<tr>
<td>10.0</td>
<td>28</td>
<td>$28f_m$</td>
</tr>
</tbody>
</table>

$^\dagger$Both upper and lower sidebands about $f_C$.

---

$\beta = 0.2$ (NBFM)

$\beta = 1.0$ (NBFM)

$\beta = 5$ (WBFM)

$\beta = 10$ (WBFM)

$B_T$ or $BW$

Single-tone Modulation Index

$$\beta = \frac{\Delta f}{f_m} = \frac{\Delta \omega}{\omega_m}$$
Spectra of FM Signals

\[ \Delta f \text{ increasing} \quad \text{&} \quad f_m \text{ is constant} \]

\[ \beta = \frac{\Delta f}{f_m} = \frac{\Delta \omega}{\omega_m} \]

\[ \Delta f \text{ is constant} \quad \text{&} \quad f_m \text{ is decreasing} \]

From A. Bruce Carlson, Communication Systems, An Introduction to Signals and Noise in Electrical Communication, 2nd edition, 1975; Chapter 6, Figure 6.5, Page 229.
Selecting an FM Station

Broadcast FM Radio covers from 88 MHz to 108 MHz
100 stations – 200 kHz spacing between FM stations

<table>
<thead>
<tr>
<th>Service Type</th>
<th>Frequency Band</th>
<th>Channel Bandwidth</th>
<th>Maximum Deviation</th>
<th>Highest Audio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commercial FM Radio Broadcast</td>
<td>88.0 to 108.0 MHz</td>
<td>200 kHz</td>
<td>±75 kHz</td>
<td>15 kHz</td>
</tr>
</tbody>
</table>

Note: 0 dBu = 0.775 volt into 600 ohms (which is equivalent to 1 mW power delivered into the 600 ohm resistor)
Measured Spectrum of an FM Radio Signal

![FM Radio Search](image)

- **IQ rate**: 10M
- **Carrier frequency**: 94.7M
- **Gain**: 15
- **Number of samples**: 200k
- **Active antenna**: RX1

Voice modulation

- **Amplitude**: -55 to -85 dB

Detected Stations:

- 93.7M
- 94.7M
- 95.5M
- 96.7M
- 98.1M
- 98.9M

**200 kHz**

**Noise**
### Specifications for Some Commercial FM Transmissions

<table>
<thead>
<tr>
<th>Service Type</th>
<th>Frequency Band</th>
<th>Channel Bandwidth</th>
<th>Maximum Deviation</th>
<th>Highest Audio</th>
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<tr>
<td>Commercial FM Radio Broadcast</td>
<td>88.0 to 108.0 MHz</td>
<td>200 kHz</td>
<td>±75 kHz</td>
<td>15 kHz</td>
</tr>
<tr>
<td>Television Sound (analog)</td>
<td>4.5 MHz above the picture carrier frequency</td>
<td>100 kHz</td>
<td>±25 kHz monaural &amp; ±50 kHz stereo</td>
<td>15 kHz</td>
</tr>
<tr>
<td>Digital TV has replaced</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Public safety – Police, Fire, Ambulance, Taxi, Forestry, Utilities, &amp; Transportation</td>
<td>50 MHz and 122 MHz to 174 MHz</td>
<td>20 kHz</td>
<td>±5 kHz</td>
<td>3 kHz</td>
</tr>
<tr>
<td>Amateur, CE class A &amp; Business band Radio</td>
<td>216 MHz to 470 MHz</td>
<td>15 kHz</td>
<td>±3 kHz</td>
<td>3 kHz</td>
</tr>
</tbody>
</table>
The Three Important Parameters in FM and PM

The three important frequencies in FM and PM are

1. Carrier frequency \( f_C \) (or \( \omega_C \))
2. Maximum modulation frequency \( f_m \) (or \( \omega_m \)), and
3. Peak frequency deviation \( \Delta f \) (or \( \Delta \omega \))

Two Definitions of importance:

1. Modulation index \( \beta \)

\[
\beta = \frac{\Delta f}{B_m} = \frac{\Delta f}{f_m} = \frac{\Delta \omega}{\omega_m} = \frac{\Delta \omega}{2\pi B_m} \quad \text{(can be a very large number)}
\]

2. Deviation ratio \( D \)

\[
D = \frac{\Delta f}{f_C} = \frac{\Delta \omega}{\omega_C} \quad \text{(always much less than unity)}
\]

Remember: For FM \( \Delta \omega = k_f m_p \) & for PM \( \Delta \omega = k_p m'_p \)
FM Bandwidth and the Modulation Index $\beta$

A. Narrowband FM (NBFM) $- \beta << 1$ radian

$$B_{FM}^{NB} \approx 2B_m \quad \text{where } B_m \text{ is the bandwidth of } m(t)$$

B. Wideband FM (WBFM) $- \beta >> 1$ radian

$$B_{FM}^{WB} \approx 2(\beta + 1)B_m \quad \text{where } \beta = \frac{\Delta f}{B_m} = \frac{\Delta f}{f_m} = \frac{\Delta \omega}{\omega_m}$$

$\Delta f$ is the peak frequency deviation $\Delta f = \max[k_f m(t)]$

$$B_{FM}^{WB} \approx 2(\Delta f + B_m) = 2(\beta + 1)B_m \quad \Leftarrow \quad \text{Carson's Rule}$$

For PM we have analogous equation,

$$B_{PM}^{WB} \approx 2(\beta + 1)B_m$$
Example: Bandwidth of FM Signal

The message signal input to a modulator is $10 \cdot \cos(2\pi \times 10^4 t)$. If frequency modulation with frequency deviation constant $k_f = 10^4 \pi$ is performed, find the bandwidth of the resulting FM signal.

Solution:

$$\beta = \frac{1}{2\pi} \cdot \frac{k_f A_m}{f_m} = \frac{10^4 \pi \times (10)}{2\pi \times 10^4} = 5$$

$$B_{FM} = 2(\beta + 1) f_m = 2(5 + 1) \cdot 10 kHz = 120 kHz,$$

using Carson's rule to calculate bandwidth $B_{FM}$.
Example: Equal Bandwidth for FM & PM Signals

If phase modulation is performed using the message signal $10 \cdot \cos(2\pi \times 10^4 t)$ used in the previous slide, find the phase deviation constant $k_p$ giving the PM signal the same bandwidth, namely, 120 kHz.

Solution:

For both the FM and PM signals to have the same bandwidth, $\beta$ and $\Delta f$ must be the same. For FM, $\Delta \omega = k_f A_m$; but for PM, $\Delta \omega = k_p m'_p$.

Expressing the message signal $m(t) = A_m \cdot \cos(\omega_m t)$ gives

$$m'(t) = \frac{d}{dt} \left( A_m \cos(\omega_m t) \right) = -\omega_m A_m \cdot \sin(\omega_m t) \quad \Rightarrow \quad m'_p = -\omega_m A_m$$

Thus,

$$k_f A_m = -k_p \omega_m A_m \quad \rightarrow \quad k_p = \frac{k_f}{\omega_m} = \frac{10^4 \pi}{2\pi \times 10^4} = \frac{1}{2}$$

Check: $\beta = \frac{k_p m'_p}{\omega_m} = \frac{k_p \omega_m A_m}{\omega_m} = k_p A_m = \frac{1}{2} (10) = 5$
Example: Commercial FM Radio Stations

For commercial FM radio, the audio message signal has a spectral range of 30 Hz to 15 kHz, and the FCC allows a frequency deviation of 75 kHz. Estimate the transmission bandwidth for commercial FM using Carson’s Rule.

Solution:

We start by calculating $\beta$

$$\beta = \frac{\Delta f}{B_m} = \frac{75 \text{ kHz}}{15 \text{ kHz}} = 5$$

Using Carson's rule gives

$$B_{FM} = 2(\beta + 1)B_m = 2(5 + 1)15 \text{ kHz} = 180 \text{ kHz}$$

The allowed bandwidth for commercial FM is 200 kHz. Note that Carson's rule slightly underestimates the bandwidth.
Why Does FM and PM Take Much More Bandwidth?

Observation: The bandwidth required for AM and NBFM are the same.

However, WBFM (wideband FM) requires much more bandwidth. Why?

A Fourier spectrum of an FM signal shows that to keep the amplitude constant of an FM signal that many components are required to represent the FM waveform. The frequency spectrum of an actual FM signal has components extending infinitely, although their amplitude decreases for sufficiently higher frequencies. Sufficiently higher frequencies applies to frequencies above the Carson bandwidth rule.

\[ B_{WB}^{FM} \approx 2(\Delta f + B_m) = 2(\beta + 1)B_m \]

\[ \Leftarrow \text{Carson's Rule} \]

Next we examine the Fourier components this using phasors.
Note: Only magnitudes of spectral lines shown.
NBFM Constructed From Phasors in FM Modulation

NBFM with tone modulation

\[ \varphi(t) = \frac{\beta A_C}{2} \]

\[ f_C \]

\[ f_m \]

\[ -f_m \]

\[ A_C \]

\[ \frac{\beta A_C}{2} \]

\[ 0 \]

\[ f_C - f_m \]

\[ f_C + f_m \]
WBFM Phasor Diagram for Arbitrary $\beta$

Sidebands Constructed From Phasors in FM Modulation

Animation showing how phase modulation works in the phasor picture -- phase modulation with a sinusoidal modulation waveform and a modulation depth of $\pi/4$ radians. The blue line segments represent the phasors at the carrier and the harmonics of the modulation frequency.
Generating FM Signals

There are two basic methods to generate FM:

1. **Direct Method** (uses voltage-controlled oscillator to vary the frequency linearly with the message signal $m(t)$)
   
   **Advantage**: Can generate large frequency deviation.
   **Disadvantage**: Carrier frequency tends to drift and must be stabilized.

2. **Armstrong’s Indirect Method** (first generate NBFM with the message signal with a small frequency deviation and then frequency multiplication is used to increase the frequency and frequency deviation to desired levels (generates WBFM))
   
   **Advantage**: More stable carrier frequency.
   **Disadvantage**: More complex hardware and cost.
Direct Generation of FM Signal Using a VCO

VCO is “voltage-controlled oscillator”

\[ \omega_{osc} \approx \frac{1}{\sqrt{LC_{eq}}} \]

\( C_{eq} \) is capacitance \( C_D \) plus capacitance of other capacitors.
Direct Generation of FM Signal Using a VCO and PLL

Input: \( m(t) \)

Mixer used as phase detector

Crystal Oscillator

\( f_{osc} = f_c / N \)

Frequency Divider \( \div N \)

VCO

\(<X\) WBFM Output Signal

Mixer used as phase detector:

\[
\cos(\omega_X t) \cdot \cos(\omega_X t + \phi) = \frac{1}{2} \left[ \cos(\omega_X t - \omega_X t - \phi) + \cos(\omega_X t + \omega_X t + \phi) \right]
\]

\[
= \frac{1}{2} \left[ \cos(-\phi) + \cos(2\omega_X t + \phi) \right] = \frac{1}{2} \left[ \cos(\phi) + \cos(2\omega_X t + \phi) \right]
\]
Narrowband FM Generated by Pulling a Crystal Oscillator

A crystal is an electro-mechanical resonator.

A crystal filter is placed in the feedback loop to stabilize the oscillator. The frequency of oscillation can be pulled slightly from the high-Q crystal resonator’s frequency. The frequency deviates only slightly and is typically only up to about 100 ppm. However, the oscillator is very stable for \( m(t) = 0 \).
Digression: Q-Values for Quartz Crystals in Electronics

A crystal oscillator is an electronic oscillator circuit that uses the mechanical resonance of a vibrating crystal of piezoelectric material to create an electrical signal with a precise frequency.

A major reason for the wide use of crystal oscillators is their high Q factor. A typical Q value for a quartz oscillator ranges from $10^4$ to $10^7$, compared to perhaps $10^2$ for an LC oscillator.

The maximum $Q$ for a high stability quartz oscillator can be estimated as $Q = 1.6 \times 10^7/f$, where $f$ is the resonant frequency in megahertz.

https://en.wikipedia.org/wiki/Crystal_oscillator
https://txccrystal.com/term.html
**Generation of Narrowband Frequency Modulation (NBFM)**

\[
\varphi_{FM}(t) = A_C \cos(\omega_C t) - A_C \left( k_f \int_{-\infty}^{t} m(\alpha) d\alpha \right) \sin(\omega_C t)
\]

NBFM is limited to \( \beta \ll 1 \) radian

---

*Agbo & Sadiku*

*Figure 4.5; page 168*
Generation of Narrowband Phase Modulation (NBPM)

\[
\varphi_{FM}(t) = A_C \cos(\omega_c t) - A_C k_p m(t) \cdot \sin(\omega_c t)
\]

Agbo & Sadiku
Figure 4.5; page 168
Indirect Generation of FM Using Frequency Multiplication

In this method, a narrowband frequency-modulated signal is first generated and then a frequency multiplier is used to increase the modulation index. The concept is shown below:

A frequency multiplier is used to increase both the carrier frequency and the modulation index by integer $N$. 
Frequency Multipliers

A frequency multiplier is a nonlinear component followed by a bandpass filter at the multiplied frequency desired.

\[ \varphi_{in}(t) \rightarrow \text{Nonlinear Device} \rightarrow y(t) \rightarrow \text{Bandpass Filter @ } n\omega_c \rightarrow \varphi_{out}(t) \]

We select the \( n^{th} \) order nonlinear component of \( y(t) \) and pass it through the bandpass filter.

\[
\varphi_{in}(t) = A_C \cdot \cos \left[ \omega_c t + k_f \int_0^t m(\lambda) d\lambda \right], \text{ and} \\
\varphi_{out}(t) = A_C \cdot \cos \left[ n\omega_c t + nk_f \int_0^t m(\lambda) d\lambda \right]
\]

Note: \( m(t) \) is not distorted by multiplier.

Conclusion: Carrier frequency is now \( nf_c \) and frequency deviation is now \( n\Delta f \).
Commercial frequency multipliers are generally \( \times 2 \) and \( \times 3 \).

Section 4.4; Page 181 of Agbo & Sadiku
Armstrong Indirect FM Transmitter Example

Crystal stabilized *voltage-controlled oscillator*

$Cf_1 = 200 \text{ kHz}$

$\Delta f_1 = 25 \text{ Hz}$

$Cf_2 = 12.8 \text{ MHz}$

$\Delta f_2 = 1.6 \text{ kHz}$

$m(t)$

NBFM generation

$\varphi_{FM}(t)$

$Cf_3 = 1.9 \text{ MHz}$

$\Delta f_3 = 1.6 \text{ kHz}$

$Cf_4 = 91.2 \text{ MHz}$

$\Delta f_4 = 76.8 \text{ kHz}$

$\varphi_{WB}(t)$

PA

$\times 48$ Multiplier

$\times 64$ Multiplier

BPF

A mixer does not change $\Delta f$

These numbers correspond to an FM broadcast radio station.

Crystal Oscillator
Why are Two Multiplication Chains Used?

\[ \phi_{FM}^{NB}(t) \quad \phi_{FM}^{WB}(t) \]

- NBFM generator
- Multiplier Chain A
- Mixer
- Multiplier Chain B
- Oscillator
Many Ways to Perform Frequency Multiplication

In electronics, a **frequency multiplier** is an electronic circuit that generates an output signal whose output frequency is a harmonic (multiple) of its input frequency. Frequency multipliers consist of a nonlinear circuit that distorts the input signal and consequently generates harmonics of the input signal.

Most multipliers are doublers or triplers
Frequency Multiplication Using Comb Generation

From our discussion on Fourier series and pulse trains:

Amplitude $V$

Comb frequencies shown

$X(f)$

Envelope: $\frac{T}{T_p} \cdot \text{sinc}(Tf)$
Simple Comb Generator

A step recovery diode (SRD) is a p-n junction diode having the ability to generate extremely short pulses. It is also called snap-off diode or charge-storage diode, and has a variety of uses in microwave electronics (e.g., pulse generator or parametric amplifier).

Comb Generator Circuit

https://www.edn.com/electronics-blogs/the-emc-blog/4402169/DIY-6-GHz-comb-generator
Step Recovery Diode Based Comb Generation

The key to generating a wide comb of frequencies is to generate very narrow pulses which step recovery diodes are designed to do.

Generation of Narrowband Phase Modulation (NBPM)

\[ \phi_{PM}(t) = A_C \cos(\omega_c t + k_p m(t)) \]

Agbo & Sadiku
Figure 4.5; page 168
Generation of Narrow Band Phase Modulation

$$m(t)$$

Carrier frequency $$f_C$$

Limitation 1: Only a small amount of phase shift is generated (low-deviation)

Limitation 2: All phase-shift circuits produce amplitude variations.

https://www.slideshare.net/sghunio/chapter06-fm-circuits
Advantages of FM

Advantages of frequency modulation

1. **Resilient to noise:** The main advantage of frequency modulation is a reduction in noise. As most noise is amplitude based, this can be removed by running the received signal through a limiter so that only frequency variations remain.

2. **Resilient to signal strength variations:** In the same way that amplitude noise can be removed, so too can signal variations due to channel degradation because it does not suffer from amplitude variations as the signal level varies. This makes FM ideal for use in mobile applications where signal levels constantly vary.

3. **Does not require linear amplifiers in the transmitter:** As only frequency changes contain the information carried, amplifiers in the transmitter need not be linear.

4. **Enables greater efficiency:** The use of non-linear amplifiers (e.g., class C and class D/E amplifiers) means that transmitter efficiency levels can be higher. This results from linear amplifiers being inherently inefficient.
Disadvantages of FM

**Disadvantages of frequency modulation**

1. **Requires a more complicated demodulator:** One of the disadvantages is that the demodulator is a more complicated, and hence more expensive than the very simple diode detectors used in AM.

2. **Sidebands extend to infinity:** The sidebands for an FM transmission theoretically extend out to infinity. To limit the bandwidth of the transmission, filters are used, and these introduce some distortion of the signal.
Ideal FM Differentiator Demodulator

The ideal FM detector converts the FM signal’s instantaneous frequency $\omega_i$ to an amplitude that is proportional to $\omega_i$.

Differentiation performs FM to AM conversion

Input: $\varphi_{FM}(t) = A_C \cos(\omega_C t + \theta(t)) = A_C \cos\left(\omega_C t + k_f \int_{-\infty}^{t} m(\lambda) d\lambda\right)$

Output: $\frac{d\varphi_{FM}(t)}{dt} = \frac{d}{dt}\left[A_C \cos\left(\omega_C t + k_f \int_{-\infty}^{t} m(\lambda) d\lambda\right)\right]$

$\frac{d\varphi_{FM}(t)}{dt} = -A_C \left(\omega_C + k_f m(t)\right) \cdot \sin\left(\omega_C t + k_f \int_{-\infty}^{t} m(\lambda) d\lambda\right)$

Both AM and FM included

After DC removal

AM allows the envelope detector to be used
Bandpass Limiter at the Receiver

For an envelope detector to work well the FM signal’s amplitude should be constant or flat. We can accomplish with a “hard limiter.” Factors such as channel noise, interference and channel fading result in amplitude variations in an FM signal’s amplitude at the receiver.

\[
x(t) = \phi_{FM}(t) = \frac{4}{\pi} \left[ \cos(\omega_c t) + k_f t \int_{-\infty}^{t} m(\lambda) d\lambda \right]
\]

From Leon W. Couch, II, Digital and Analog Communication Systems, 8th edition, 2013; Figure 4-7 (page 265).
**Practical FM Differentiator Demodulator**

![Diagram of a differentiator circuit](image)

**Differentiator at low frequencies**

The high-pass filter in the frequency domain is equivalent to differentiation in the time domain! The high-pass filter acts as a differentiator for an FM signal. Therefore,

\[ H(j\omega) = \frac{j \omega RC}{1 + j \omega RC} = \frac{j (\omega / \omega_{3dB})}{1 + j (\omega / \omega_{3dB})}; \text{ where } \omega_{3dB} = \frac{1}{RC} \]

For \( \omega \ll \omega_{3dB} = \frac{1}{RC} \); then \( H(j\omega) \approx j\omega RC \)

Multiplication by \( j\omega \) in the frequency domain is equivalent to differentiation in the time domain!

**Envelope Detector**

\[ y(t) = A_c \omega_c RC + A_c \omega_c RC k_f m(t) \]

Envelope detector extracts \( m(t) \)
Bode Plot of CR High-Pass Filter

Bode Diagram

Amplitude (dB)

-40  -35  -30  -25  -20  -15  -10  -5  0

20 dB /decade

ω_{3dB}

Normalized frequency

Normalized frequency

Phase (degrees)

90  80  70  60  50  40  30  20  10  0

45°
Practical Frequency Demodulators

Frequency discriminators can be built in various ways:

- Time-delay demodulator (uses differentiation)
- FM slope detector (FM to AM conversion)
- Balanced discriminator
- Quadrature demodulators
- Phase locked loops (a superior technique)
- Zero crossing detector
This is an implementation of discrete time approximation to differentiation.

$$y(t) = \frac{1}{\tau} \left( \phi_{FM}(t) - \phi_{FM}(t - \tau) \right)$$

$$\frac{d\phi_{FM}(t)}{dt} = \lim_{\tau \to 0} y(t) = \lim_{\tau \to 0} \left[ \frac{1}{\tau} \left( \phi_{FM}(t) - \phi_{FM}(t - \tau) \right) \right]$$

It can be shown that an adequate value for $\tau$ is less than $T/4$, where $T$ is the period of the unmodulated carrier for the FM signal. Again, this relies upon FM to AM conversion after which the envelope detector recovers $m(t)$. 
An FM Slope Detector Performs FM to AM Conversion

Comment: The differentiation operation is performed by any circuit acting as a frequency-to-amplitude converter.

Operates on the skirt of the LC resonance curve
Balanced Discriminator (Foster-Seeley Discriminator) – 1936

Another example of the use of symmetry in design.

Transfer Characteristic

Centered around $f_c$
Quadrature Demodulator – Block Diagram

FM signal is converted into PM signal

PM signal is used to recover the message signal $m(t)$

Signal delay $\tau_0$ times carrier frequency $f_c$ = 90 degrees (or $\pi/2$).
Using a XOR Gate for Phase-Frequency Detector

Purpose: To produce a signal current or voltage, proportional to the difference in phase or frequency between two input signals.

\[ V_{\text{out}} = K_D \Delta \phi \]

Example

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Exclusive OR gate
Quadrature Demodulator – Implementation

The signal is split into two components. One passes through a network providing a basic 90° phase shift in addition to the phase shift from the signal’s frequency deviation. The mixer output is dependent upon the phase difference between the two signals; that is, it acts as a phase detector producing a voltage output proportional to the phase difference and thus the frequency deviation on the FM signal.

Phase-Locked Loops (Using Feedback)

A PLL consists of three basic components:
- Phase detector
- Loop filter
- Voltage-controlled oscillator (VCO)

 PLL Diagram:

\[ A_C \left[ \cos(\omega_C t + \theta_i(t)) \right] \]

\[ 2A_{VCO} \left[ \cos(\omega_C t + \theta_o(t)) \right] \]

Output signal is phase difference
Zero-Crossing Detectors

• Zero-Crossing Detectors are also used because of advances in digital integrated circuits.

• These are the frequency counters designed to measure the instantaneous frequency by the number of zero crossings.

• The rate of zero crossings is equal to the instantaneous frequency of the input signal

An example is shown on the next slide.
Zero-Crossing Detector Illustration

Zero crossing detector

\[ \varphi_{FM}(t) \rightarrow \text{Hard Limiter} \rightarrow \text{Zero-crossing circuit} \rightarrow \text{Multi-vibrator} \rightarrow \text{Averaging circuit} \rightarrow m(t) \]

More frequent ZC’s gives higher instantaneous frequency which causes greater average signal.

https://www.slideshare.net/avocado1111/angle-modulation-35636989
Noise in Frequency Modulation

In FM systems noise has a greater effect on the higher modulating frequencies. It is common practice to boost the signal level of the higher modulating frequencies to improve the signal-to-noise ratio of the overall transmitted FM signal.

**Differentiator Demodulator (Slides 91 & 93)**

This artificial boosting at the transmitter is called “pre-emphasis” and the removal of the boost at the receiver is called “de-emphasis.”

The result is an improvement in the discernible quality of received FM signals.

\[
S_{N_o}(f) \approx N_O \left(2\pi f\right)^2 \text{ for } |f| < \frac{B_T}{2}
\]

Power Spectral Density (PSD) of output noise in an FM receiver.

(Increases because noise is differentiated in FM receiver)

**Fig. 10.9; p. 578 of 4th ed., Lathi & Ding**
Pre-Emphasis and De-Emphasis in FM

Channel noise acts as interference in FM and is uniform over the entire BW. Voice and music have more energy at lower frequencies, so we need to “emphasize“ their upper frequencies by filtering. However, the HF emphasis must be removed at the receiver using a “de-emphasis” filter.

(Widely used commercially in the recording industry)

Filtering improves SNR in FM transmission.
Typical Pre-Emphasis and De-Emphasis Filters

<table>
<thead>
<tr>
<th>Transmitter</th>
<th>Receiver</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Pre-emphasis Filter</strong></td>
<td><strong>De-emphasis Filter</strong></td>
</tr>
<tr>
<td><img src="image" alt="Pre-emphasis Filter Diagram" /></td>
<td><img src="image" alt="De-emphasis Filter Diagram" /></td>
</tr>
<tr>
<td>[ H(\omega) = \frac{V_{out}}{V_{in}} = \frac{1 + j\omega R_1 C}{1 + j\omega (R_1 \parallel R_2) C} ]</td>
<td>[ H(\omega) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + j\omega R_1 C} ]</td>
</tr>
<tr>
<td>[</td>
<td>H(\omega)</td>
</tr>
<tr>
<td><img src="image" alt="Graph" /></td>
<td><img src="image" alt="Graph" /></td>
</tr>
<tr>
<td>+6 dB/octave</td>
<td>-6 dB/octave</td>
</tr>
<tr>
<td>2.1 kHz (\frac{1}{R_1 C})</td>
<td>2.1 kHz (\frac{1}{R_1 C})</td>
</tr>
<tr>
<td>33 kHz (\frac{1}{(R_1 \parallel R_2) C})</td>
<td>33 kHz (\frac{1}{(R_1 \parallel R_2) C})</td>
</tr>
</tbody>
</table>
Analog and Digital FM Cellular Telephones

1G analog cellular telephone (1983) – AMPS (Advanced Mobile Phone Service)
First use of cellular concept . . .
Used 30 kHz channel spacing (but voice BW was B = 3 kHz)
  Peak frequency deviation $\Delta f = 12$ kHz, and
  $B_T = 2(\Delta f + B) = 2(12$ kHz $+ 3$ kHz $) = 30$ kHz
Two channels (30 kHz each); one for uplink and one for downlink
Used FM for voice and FSK (next slide) for data communication
No protection from eavesdroppers!

Successor to AMPS was GSM (Global System for Mobile) in early 1990s
GSM is 2G cellular telephone
Still used by nearly 50% of world’s population (as of 2017)
GSM was a digital communication system
  Modulating signal is a bit stream representing voice signal
Uses Gaussian Minimum Shift Keying (GMSK)
Channel bandwidth is 200 kHz (simultaneously shared by 32 users
  This is 4.8 times improvement over AMPS

More to come on cellular . . .
Digital Carrier Modulation – ASK, FSK and PSK

$m(t)$

Amplitude Shift Keying

Frequency Shift Keying

Phase Shift Keying

$k_p m_d(t) \sim (-\pi, \pi)$

Fig. 5-16  Digital carrier modulation

https://slideplayer.com/slide/12711804/
Digital Phase Shift Modulation

Binary Phase Shift Keying (BPSK) \( k_p m_d(t) \sim (-\pi, \pi) \)

The wave shape is ‘symmetrical’ at each phase transition because the bit rate is a sub-multiple of the carrier frequency \( \omega_c/(2\pi) \). In addition, the message transitions are timed to occur at the zero-crossings of the carrier.
Questions?

Triangular-Wave FM Generation

Switching-Circuit Phase Modulator

FM System Improvement in SNR

The signal-to-noise ratio (SNR) improvement in an FM system is a function of modulation index $\beta$,

$$SNR_{FM} = 3\beta^3(\beta + 1) \cdot CNR,$$

where $CNR$ is carrier-to-noise ratio

$$SNR_{FM} = 3\left(\frac{B_T}{2B_m}\right) \cdot CNR$$

Example: For FM transmission bandwidth $B_T$ of 200 kHz and a message bandwidth $B_m$ of 15 kHz ($\beta = 5.67$), the improvement in the SNR at the output of an FM receiver to have an FM gain of 27 dB above the CNR.

This is essentially a tradeoff between message signal quality (SNR) and FM transmission bandwidth. Thus, greater transmission bandwidth is the key to FM’s superior performance.
PSD of AWG Noise Through Differentiator Network

PSD of input noise (Uniform White Noise): $S_{N_i}(f) = K$

The transfer function of a differentiator is given by $H(f) = j2\pi f$

The PSD of the output noise is calculated by

$$S_{N_o}(f) = |H(f)|^2 \times S_{N_i}(f) = \left|j2\pi f\right|^2 \times K,$$

Therefore,

$$N_o = \int_{-B}^{B} \left|j2\pi f\right|^2 \times K \, df = \frac{8\pi^2 K}{3} B^3 \text{ watts}$$
FM System As Special Case of PM System

After: Lathi & Ding, 4th ed.; page 577

\[
S_o \quad N_o \quad S_o \quad N_o = 3 \left( \frac{k_f^2 \langle m^2 \rangle}{(2\pi B)^2} \right) \times \left( \frac{A_c^2}{2N} \right)
\]

where \( N = \frac{A_c^2 N_o}{2B} \)
Phase Modulator Circuit

Limitation 1: Only a small amount of phase shift is generated (low-deviation)
Limitation 2: All phase-shift circuits produce amplitude variations.