Problem 1 High-Q Resonant RLC Circuit (30 points)

One possible frequency selective circuit is a simple LRC resonator as schematically shown below. The resonant frequency of this circuit is given by

\[ f_{\text{resonance}} = \frac{1}{\sqrt{LC}} \]

The Quality factor of a resonant circuit (Q-factor for short) is defined as the resonant frequency divided by the half-power (or -3 dB) bandwidth; in symbols it is

\[ Q = \frac{f_{\text{resonance}}}{B} \]

An RLC circuit finds use in selecting communication bands by tuning to the desired carrier frequency. It is also used for “demodulation” of amplitude modulation (AM) communication signals, “modulation” of frequency modulation (FM) communication signals and for establishing oscillation frequencies in local oscillators used in superheterodyne receivers. Consider the parallel RLC circuit as shown below:

(a) Derive the transfer function \( H(2\pi f) \) for this parallel RLC circuit. Assume the sinusoidal steady-state in deriving the transfer function. We define \( H(2\pi f) \) as the ratio of the current \( i_R \) flowing through the resistor divided by the input current \( i(t) \).

Answer:

\[ H(\omega) = \frac{i_R}{i(t)} = \frac{j\omega L}{R + j\omega L + (j\omega)^2 RCL} \]
(b) Derive the half-power bandwidth $B$ (i.e., the -3 dB bandwidth forming the frequency band between the -3 dB frequencies) for this circuit. Express bandwidth $B$ as a function of $R$, $L$ and $C$.

**Answer:**

\[
\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + (LC)^2} \quad \text{and} \quad \omega_2 = +\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + (LC)^2}
\]

Bandwidth $B = \omega_2 - \omega_1 = \frac{1}{RC}$

(c) Starting from the definition of $Q$-factor at the beginning of the problem statement; derive an expression for $Q$ as a function of the circuit parameters.

**Answer:**

\[
\text{Q-factor: } Q = \frac{\omega_0}{B} = \frac{\omega_0 RC}{\omega_0 L} \quad \text{where} \quad \omega_0 = \frac{1}{\sqrt{LC}}
\]

**Problem 2 Distortionless Networks** (10 points)

In Lathi and Ding, Section 3.4.2 on pages 125 to 127, distortionless transmission is discussed. A distortionless network has the general characteristic as shown below:

In words, distortionless transmission requires the magnitude of the transfer function $|H(f)|$ to be constant over all frequencies and the phase $\phi_h(f)$ variation from transmission through the network to be linear (i.e., proportional to the frequency $f$).
Suppose you have a length of an ideal coaxial cable. The characteristics shown by the faint lines on the plot below represent a one foot long section of this ideal coaxial cable. Next, you are given a two-foot long section of this coaxial cable. Sketch the change in magnitude and phase for this two-foot long section of the coaxial cable.

**ANSWER:**

![Graph showing change in magnitude and phase](image)

**Problem 3 White Filtered Noise (30 points)**

Noise is a random signal (also random noise power) without information content. In other words, it is unwanted, but nature provides noise power for free in electronic circuits. A “white noise model” is useful in analyzing the effect of noise on communication systems and also for characterizing separate communication components. For example, filters are extensively used in communication systems for many reasons. In this problem we study white filtered noise in a simple low-pass $RC$ filter.

**Background:**

White noise has a constant noise power per unit bandwidth (watts/Hz) at all useful frequencies. In other words, it has a constant “Power Spectral Density” $G_n(f)$ (aka PSD) for all frequencies $f$. This is illustrated in the single-sided PSD plot for white noise below. [Note: A two-sided PSD would also show negative frequencies.]
$N_0$ is the constant magnitude of the noise power (in watts/Hz) at a node in the communication system. At any node the PSD is proportional to the mean-square of the voltage per hertz or the mean-square current per hertz. The implication of a constant PSD is that the total noise power $P_t$ over a bandwidth of $B$ hertz is given by $P_t = N_0 \times B$ watts. Inserting a filter changes the total noise power because the filter shapes the frequency response of the system.

We want to calculate the total noise power $P_t$ output with the $RC$ low-pass filter inserted. The input noise source is white noise which is proportional to the mean-squared value of the noise voltage, namely $\langle e_{n, in}^2 \rangle$, and we assume a normalized impedance of one ohm for convenience (that restriction is easily removed by putting in the correct resistance). Assume identical input and output impedances for the filter. Therefore, $\langle e_{n, in}^2 \rangle$ is proportional to the input noise power and $\langle e_{n, out}^2 \rangle$ is proportional to the output noise power from the filter driven by $\langle e_{n, in}^2 \rangle$.

The filter attenuates the noise power at frequencies above its cutoff frequency (i.e., its 3-dB frequency given by $f_{-3dB} = 1/(2\pi RC)$). Hence, we can write,

$$\langle e_{n, out}^2 \rangle = |H(f)|^2 \cdot \langle e_{n, in}^2 \rangle$$

(a) Find $\langle e_{n, out}^2 \rangle$ in terms of $\langle e_{n, in}^2 \rangle$ when the filter is inserted. [Hint: use the following definite integral in evaluating your expression for the total noise power out.]

From a mathematics table of definite integrals:

$$\int_0^\infty \frac{adx}{a^2 + x^2} = \frac{\pi}{2} \quad (\text{for} \quad a > 0)$$
ANSWER:

We start with finding $H(f)$; for an $RC$ low-pass filter as shown,

$$H(f) = \frac{1}{1 + j(\omega / \omega_{-3dB})} \quad \text{where } \omega_{-3dB} = 2\pi f_{-3dB} = \frac{1}{RC} = \frac{1}{\tau_{RC}}$$

Therefore,

$$|H(f)|^2 = \frac{1}{(1 + \left(\frac{2\pi f_{-3dB}}{2\pi f_{-3dB}}\right))^2} = \frac{1}{1 + \left(\frac{\omega}{\omega_{-3dB}}\right)^2}$$

$$|H(f)|^2 = \frac{(\omega_{-3dB})^2}{(\omega_{-3dB})^2 + (\omega)^2} = \frac{\left(\frac{1}{RC}\right) \left(\frac{1}{RC}\right)}{\left(\frac{1}{RC}\right) + \omega^2} = \left(\frac{1}{RC}\right) \frac{a}{a^2 + \omega^2}$$

We have set $a = \frac{1}{RC}$; and $2\pi df = d\omega$. Using, $\langle e_{n,\text{out}}^2 \rangle = |H(f)|^2 \cdot \langle e_{n,\text{in}}^2 \rangle$

$$\langle e_{n,\text{out}}^2 \rangle = |H(f)|^2 \cdot \langle e_{n,\text{in}}^2 \rangle = \langle e_{n,\text{in}}^2 \rangle \times \left(\frac{1}{RC}\right) \int_0^{\infty} \frac{a d\omega}{a^2 + \omega^2}, \quad \text{where } a = \left(\frac{1}{RC}\right)$$

$$\langle e_{n,\text{out}}^2 \rangle = \langle e_{n,\text{in}}^2 \rangle \times \left(\frac{1}{RC}\right) \times \frac{\pi}{2} = \frac{\pi}{2} \omega_{-3dB} \langle e_{n,\text{in}}^2 \rangle$$

(b) We can define an effective noise bandwidth $B_{\text{eff}}$ for a filter relative to its half-power bandwidth, denoted by $B_{3dB}$. What is $B_{\text{eff}}$ for this $RC$ low-pass filter?

$$B_{\text{eff}} = \frac{\pi}{2} B_{3dB}$$

Problem 4 problem (30 points)

We are given a high-pass filter as shown below which we drive with a periodic square waveform. The high-pass filter and its transfer function are as given below:

$$H(j\omega) = \frac{v_{\text{out}}(t)}{v_{\text{in}}(t)} = \frac{j\omega}{1 + j(\omega / \omega_{-3dB})}; \quad \omega_{-3dB} = \frac{1}{RC}$$
The input square wave is shown here (note it is centered on the origin making it an even function – that is why its Fourier series is in terms of cosines rather than sines).

\[ v_{in}(t) = \frac{4A}{\pi} \left[ \cos(\omega t) - \frac{1}{3}\cos(3\omega t) + \frac{1}{5}\cos(5\omega t) - \ldots \right] \]

**Problem 4 statement:**

Two students are queried about the effect of passing the square-wave signal through the \( RC \) high-pass filter. They were told that the 3-dB break frequency, denoted by \( f_{3dB} \), is four octaves below the fundamental frequency of the square-wave input signal. The square-wave fundamental frequency is given by \( f = 1/T \), where \( T \) is the period. [Note: An octave is a factor of 2, or \( \frac{1}{2} \); so four octaves is either \( 2 \times 2 \times 2 \times 2 = 16 \) or \( \frac{1}{16} \), depending upon if we are talking about being above or below.]

**Tom’s argument:** Tom argues that the because the fundamental frequency of the square wave is 16 times greater than the corner frequency of the high-pass filter, and all of the square wave’s harmonic frequencies are even higher in frequency, the square-wave signal will pass directly as applied at the input directly to the output. It will be almost as if the high-pass filter is not even present. Do you agree with Tom’s argument? Why?

**ANSWER:** Tom’s argument is correct, but he forgot about any DC component needed to sustain the flat top of the square-wave waveform.

**Matt’s argument:** Matt doesn’t think this is the whole story and that the square-wave signal will not pass through the without being distorted. But he is not quite sure if this is true? So Matt does what any good engineering student would do – he performs an experiment! He builds the filter and uses a function generator to drive its input with a square waveform. The output waveform observed by Matt appears below:
Obviously, Matt’s suspicion is correct. Yes, the fundamental frequency and all its harmonics are far above the -3-dB frequency of the high-pass filter. But something else is going on. Explain why the tops of the square wave are sloped as shown.

ANSWER:

Whereas the frequency components making up the periodic square wave do pass through the high-pass filter, the voltage across the capacitor $C$ when attempting to hold a constant value will sag toward zero voltage. The reason for this behavior is that for the capacitor to maintain a constant voltage, it must retain all of the charge maintaining this voltage level (remember $C = Q/V$). The charge on the capacitor can discharge through the resistor $R$ and it decays with time constant $\tau = RC$. The voltage supply allows the discharge current to flow.

Another viewpoint: The series capacitor acts as a differentiator to the input signal. This means that it responds to time varying changes in the input signal but does not hold DC signal levels because the derivative of a constant is zero.