Problem 1 Relative Refractive Index Difference  (15 points)

Equation (2.9) on page 19 of Senior, 3rd edition, defines the relative refractive index difference as

\[
\Delta \triangleq \frac{n_1^2 - n_2^2}{2n_1^2} \simeq \frac{n_1 - n_2}{n_1}
\]

(a) Derive (or show a proof of) the approximation to the definition of \( \Delta \) (the approximation is the right hand term of the above equation).

\[
\Delta \triangleq \frac{n_1^2 - n_2^2}{2n_1^2} = \frac{(n_1 + n_2)(n_1 - n_2)}{2(n_1 \times n_1)}
\]

But \( n_1 \) and \( n_2 \) are approximately equal, so \( (n_1 + n_2) \simeq 2n_1 \)

\[
\therefore \Delta \simeq \frac{n_1 - n_2}{n_1} \text{ as was to be shown.}
\]

(b) If \( n_1 = 1.500 \) and \( n_2 = 1.485 \), then calculate both expressions for \( \Delta \) and compare them with respect to their values.

\[
\Delta \triangleq \frac{n_1^2 - n_2^2}{2n_1^2} = \frac{(1.500)^2 - (1.485)^2}{2(1.500)^2} = \frac{0.044775}{4.50000} = 0.009950
\]

\[
\Delta \simeq \frac{n_1 - n_2}{n_1} = \frac{1.500 - 1.485}{1.500} = 0.010000
\]

Discussion:

These two numbers are very close to each (actually they are within one-half of one percent of each other). So this is a good approximation so long as the two refractive indices are close in magnitude to each other (say less than two percent).
**Problem 2 Numerical Aperature and Acceptance Angle** (15 points)

Consider a silica optical fiber that is a step index fiber. The velocity of light in the fiber is \(2.01 \times 10^8\) meters/second and the critical angle at the core-to-cladding interface is 80 degrees. Determine (1) the numerical aperture (NA) and (2) the acceptance angle (\(\alpha\)) for the fiber in air \((n_o)\). Assume this fiber has a core diameter suitable for being analyzed using a ray analysis. Take the velocity of light in vacuum to be \(3.00 \times 10^8\) meters/second.

**Solution:**

Using eq. (2.39) on page 30 in Senior (3rd ed.), the velocity is

\[
\nu = \frac{c}{n_1}\; \text{and} \; n_1 = \frac{3.00 \times 10^8}{2.01 \times 10^8} = 1.492
\]

Using eq. (2.2) for the critical angle \(\theta_{cr}\) we get

\[
\sin(\theta_{cr}) = \frac{n_2}{n_1}, \; \text{then} \; n_2 = 1.492 \times \sin(80^\circ) = 1.469
\]

The NA is given by eq. (2.8) on page 18 in Senior (3rd ed.), and assume \(n_o\) of air is unity (it actually is 1.0003 in value) for simplicity, then we have

\[
NA = n_o \sin(\alpha) = \sqrt{(n_1)^2 - (n_2)^2}
\]

\[
NA = \sin(\alpha) = \sqrt{(1.492)^2 - (1.469)^2}
\]

\[
NA = \sqrt{0.0681} = 0.261 \quad \Leftarrow
\]

and the acceptance angle is (assuming \(n_o = 1.0000\))

\[
\alpha = \sin^{-1}(NA) = \sin^{-1}(0.0261) = 15.13^\circ \quad \Leftarrow
\]

Sometimes the acceptance angle is taken to be \(2\alpha\)

Thus, \(\theta_{ac} = 2\alpha = 30.26^\circ \quad \Leftarrow\)

**Problem 3 Estimating the Speed of Light in a Fiber** (20 points)

A step index fiber has a solid acceptance angle of 0.115 steradians and a relative refractive index difference of \(\Delta = 0.9\%\). Estimate the speed of light in the optical fiber.
**Hint:** First you might want to find the relationship between solid angle and the acceptance angle for small angles.

**Solution:**

First we find the relationship between the solid angle and acceptance angle from the figure below: [https://www.quora.com/What-is-solid-angle](https://www.quora.com/What-is-solid-angle)

From this figure the solid angle $\Omega$ is

$$\Omega = (4\pi) \left( \frac{\pi r^2}{4\pi R^2} \right) = \pi \left( \frac{r^2}{R^2} \right) \text{[steradians]}$$

Then using the small angle approximation we note that

$$\sin(\alpha) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{r}{R} \quad (\theta = \alpha)$$

and we can write

$$\text{NA} = n_0 \sin(\alpha) = \sin(\alpha) = \sqrt{(n_1)^2 - (n_2)^2}$$

assuming that $n_0$ is unity. From

$$\Omega = \sin(\alpha) = \pi (\text{NA})^2, \text{ thus}$$

$$\left(\frac{\text{NA}}{\pi}\right)^2 = \frac{\Omega}{\pi} = \frac{0.115}{3.1416} = 0.0366$$

$$\therefore \text{NA} = 0.1913$$

Using eq. (2.10), the NA is also is given by

$$\text{NA} = \frac{\text{NA}}{\sqrt{2\Delta}}$$

$$n_1 = \frac{\text{NA}}{\sqrt{2\Delta}} = \frac{0.1913}{\sqrt{2(0.009)}} = 1.426$$
The speed of light in the core is
\[ \nu = \frac{c}{n_1} = \frac{3.000 \times 10^8}{1.426} = 2.104 \times 10^8 \text{ m/sec} \]

**Problem 4  Core Diameter in MMF  (10 points)**

A multimode step index fiber has a relative refractive difference of 1% and a core index of refraction of 1.500. The number of modes propagating in the fiber at the wavelength \( \lambda = 1300 \text{ nm} \) is 1100. Estimate the diameter \( d \) of the optical fiber.

**Solution:**

Using eq. (2.74) on page 45, the mode volume \( M \) is given by
\[ M = \frac{V^2}{2}; \quad \text{so} \quad V^2 = 2 \times M = 2 \times 1100 = 2200 \]

\[ \therefore V = \sqrt{2200} = 46.90 \]

Then with eq. (2.70) we can determine the core radius \( a \), thus
\[ V = \frac{2\pi}{\lambda} an_1 \sqrt{2\Delta}; \quad \text{thus} \quad a = \frac{V\lambda}{2\pi n_1 \sqrt{2\Delta}} \]
\[ a = \frac{(46.90)(1.3 \times 10^{-6})}{2\pi(1.5) \times \sqrt{0.02}} = 45.7 \ \mu\text{m} \]

Diameter \( d = 2a = 91.5 \ \mu\text{m} \)

**Problem 5  Multimode Graded Index Fiber  (10 points)**

A multimode graded index fiber has an acceptance angle of 8\(^\circ\) in air. Estimate the relative refractive index difference \( \Delta \) between the core and the cladding when the refractive index of the core \( n_1 = 1.62 \). Assume the index of refraction of air is unity.

**Solution:**

From eqs. (2.8) and (2.10) on pages 18 and 19, we can solve for \( \Delta \) as follows:
NA = \( n_0 \sin(\alpha) = (1) \cdot \sin(8^\circ) = 0.139 \)

\[ NA = n_1 \sqrt{2\Delta} \quad \text{or} \quad \Delta = \frac{1}{2} \left( \frac{NA}{n_1} \right)^2 \]

\[ \Delta = 0.5 \left( \frac{(0.139)^2}{(1.62)^2} \right) = 3.690 \times 10^{-3} = 0.37\% \quad \Leftarrow \]

**Problem 6 Cutoff Frequency of Fiber**  
(15 points)

We have a step index optical fiber with a core diameter of 8 micrometers and the indices of refraction are \( n_1 = 1.500 \) and \( n_2 = 1.497 \) for the core and cladding, respectively. Calculate the cut-off wavelength of the optical fiber (where the operating wavelength \( \lambda > \lambda_{\text{cutoff}} \)).

**Solution:**

We will use equation (2.98) for the cut-off wavelength,

\[ \lambda_{\text{cutoff}} = \frac{\pi (2a)}{V_c} n_1 \sqrt{2\Delta}, \quad \text{where} \quad NA = n_1 \sqrt{2\Delta} \]

Also, \( NA = \sqrt{n_1^2 - n_2^2} = \sqrt{(1.500)^2 - (1.497)^2} = 0.948 \)

Then, \( \lambda_{\text{cutoff}} = \frac{\pi (8 \times 10^{-6})}{2.405} \times 0.948 = 0.991 \mu m \quad \Leftarrow \)
Problem 7 Fiber-Optic System (15 points)

Consider an optical fiber system as shown in the drawing below.

The loss of the fiber link is 0.28 dB/km and its length is \( L = 80 \) km. The loss of the optical filter is -1.2 dB and the gain of the amplifier (block \( G_3 \)) is +17 dB. Suppose that the receiver must be presented with a power level of at least \(-3\) dBm to operate without bit errors. **Calculate the minimum transmitter power** \( P_{\text{in}} \) **to operate this fiber-optic system.** **Express your answer for** \( P_{\text{in}} \) **in both “dBm” and in “milliwatts”**.

Solution:

Fiber link loss: \( F_1 \) (dB) = \(-0.28 \) dB/km \( \times \) 80 km = \(-22.4\) dB

Filter loss: \( F_2 \) = - 1.2 dB

Amplifier gain: \( G_3 \) = 17 dB

The minimum power needed at the receiver is \( P_{\text{out}} = -3\) dBm

\[
P_{\text{out}} = P_{\text{in}} + F_1 + F_2 + G_3
\]

\(-3\) dBm = \( P_{\text{in}} - 22.4\) dB - 1.2 dB + 17 dB

\( P_{\text{in}} \) is the transmitter power which is \( P_{\text{in}} = +3.6\) dBm

\[
P(\text{mW}) = 1\ \text{mW} \times 10^{(P_{\text{in}}(\text{dBm})/10)} = 1\ \text{mW} \times 10^{(3.6/10)} = 2.291\ \text{mW}
\]