EE 443 Homework #4 Solutions  
(Fall 2020 – Due September 29, 2020)  
Print out homework and do work on the printed pages.  
Total of 100 points


Problem 1 Einstein B Coefficient (20 points)

For an atomic system at T = 300 Kelvin, the two-level spontaneous lifetime associated with a 2 → 1 transition is $\tau_{21} = 1.5$ nanoseconds. The energy difference between the two levels is $2.4 \times 10^{-19}$ joule. **Calculate the Einstein $A_{21}$ and $B_{21}$ coefficients**, assuming the velocity of light in the medium to be $3 \times 10^8$ meters/second. Also, assume the degeneracies of both levels to be equal and unity. [Note: Planck’s constant is $6.626 \times 10^{-34}$ joule-second and Boltzmann’s constant is $1.381 \times 10^{-23}$ joule/Kelvin.]

Finally, don’t forget to specify the correct units for the Einstein $A_{21}$ and $B_{21}$ coefficients.

Solution:

Start by calculating the $A_{21}$ coefficient using the lifetime $\tau$ (Lecture 7, slide 11)

$$A_{21} = \frac{1}{\tau_{21}} = \frac{1}{1.5 \times 10^{-9} \text{[sec]}} = 6.667 \times 10^8 \text{ sec}^{-1} \text{ or } 666.67 \text{ MHz}$$

Also, the frequency is determined from

$$\Delta E = hf \quad \text{thus,} \quad f = \frac{2.4 \times 10^{-19} \text{[J]}}{6.626 \times 10^{-34} \text{[J-sec]}} = 3.622 \times 10^{14} \text{ Hz}$$

To calculate the $B_{21}$ coefficient we use (from Lecture 7, slide 13)

$$\frac{A_{21}}{B_{21}} = \frac{8\pi hf^3}{c^3} \quad \text{thus,} \quad B_{21} = \frac{A_{21}c^3}{8\pi hf^3}$$

Substituting values gives,
Problem 2  Naperian vs. decadic attenuation coefficients  (10 points)

The attenuation coefficient characterizes how easily a volume of material can be penetrated by a beam of light, sound, particles, or other energy or matter.

Just as the Naperian attenuation coefficient $\alpha_e$ measures the number of e-fold reductions that occur over a unit length of material, the decadic attenuation coefficient $\alpha_{10}$ measures how many 10-fold reductions occur. In other words, a decadic attenuation coefficient of 1 m$^{-1}$ means 1 m of material reduces the penetrating radiation by a factor of 10. What is the numerical relationship between $\alpha_e$ and $\alpha_{10}$?

**Answer:**

Equating $e^{-\alpha_e x} = 10^{-\alpha_{10} x}$, then $\ln(e^{-\alpha_e x}) = \ln(10^{-\alpha_{10} x})$

Note: $\ln(10) = 2.3026$ and $\log_{10}(e) = 0.4343 = \frac{1}{2.3026}$

$-\alpha_e \cdot \ln(e) = -\alpha_{10} \cdot \ln(10)$, thus we find (since $\ln(e) = 1$)

$\therefore \alpha_e = 2.3026 \cdot \alpha_{10}$  

and $\alpha_{10} = \frac{\alpha_e}{\ln(10)} = 0.4343 \cdot \alpha_e$  

Problem 3  Longitudinal Mode Spacing in Cavity  (10 points)

A Fabry-Pérot cavity semiconductor laser has the following properties:

Internal loss of 70 cm$^{-1}$, mirror reflectivity of $R_1 = R_2 = 0.36$, and the distance between the partially reflecting mirrors is 500 micrometers (i.e.,
500 \mu m). The index of refraction $n$ of the cavity material = 3.3. **Calculate the longitudinal mode spacing $\Delta f$.**

**Solution:**

We begin with the longitudinal mode spacings (using Lecture 7, slide 25),

$$\Delta f = \frac{c}{2nL} = \frac{3 \times 10^8 \text{ m/} \text{sec}}{2(3.3)(500 \times 10^{-6} \text{ m})}$$

$$\Delta f = 9.091 \times 10^{10} \text{ Hz or 90.91 GHz} \iff$$

**Problem 4 Cavity Loss Estimation** (15 points)

The coated mirror reflectivities at the ends of a 350 micron long optical cavity of an injection laser are 0.5 and 0.65. At normal operating temperature the threshold current density for the device is 2000 Ampere/cm$^2$ and the gain factor $\beta = 0.022$ cm/Ampere. Estimate the loss coefficient of the optical cavity and express it in units of 1/cm.

**Solution:**

We begin with equation (6.34) on page 3.22 and note that the cavity length $L$ is to be expressed in centimeters, thus, $L = 3.50 \times 10^{-2} \text{ cm}$.

$$J_{th} = \frac{1}{\beta} \left[ \tilde{\alpha} + \frac{1}{2L} \log_e \left( \frac{1}{r_1 r_2} \right) \right]$$

$$\therefore \quad \tilde{\alpha} = \beta J_{th} - \frac{1}{2L} \log_e \left( \frac{1}{r_1 r_2} \right)$$

$$\tilde{\alpha} = (0.022)(2000) - \frac{\ln(1/0.325)}{2(0.035)}$$

$$\tilde{\alpha} = 44 - 16.1 = 27.9 \text{ cm}^{-1} \iff$$
Problem 5  Optical Fiber Link Power Budget  (15 points)

We are given a design problem for a simple optical fiber link. The source LED is capable of emitting +6 dBm coupled into the optical fiber. There are four connectors used to connect the fiber to the transmitter, amplifier and photodiode detector (receiver). The connectors each have 0.7 dB loss per connector. The amplifier has +16 dB of optical gain. The photodiode has a minimum sensitivity (threshold of signal detection) of -44 dBm. The type of optical fiber used in the link has a loss of 0.58 dB/km and both sections (of length L1 and length L2) must use the same optical fiber type. We also want to allow for an 8 dB margin in the fiber link design to allow for component variability, temperature effects, and aging. Length L1 is 30 kilometers (km) and we are asked to determine how long length L2 can be to meet the design parameters stated above. The figure below shows the link.

Optical Fiber Link

Find length L2 for the second fiber connecting to the detector.

Solution:

The losses and gains are as follows:
Loss from four connectors = 2.8 dB = (0.7 dB/conn \times 4 \text{ conn})
Loss from L1 length of fiber = 17.4 dB = (0.58 dB/km \times 30 \text{ km})
Loss from L2 length of fiber is the unknown (To Be Determined)
Gain from amplifier = 16 dB

The allowable loss in the link is determined from the difference between the input power and the minimum detectable power.
minus the required margin of + 8 dB. This is +6 dbm – (-44 dBm) – 8 dB margin = 42 dB allowable loss in the entire link. Then we write the equation:

Total allowable loss = connector loss + L1 loss + L2 loss – Gain
Total allowable loss = 2.8 dB + 17.4 dB + L2 loss – 16 dB = 42 dB
So the L2 loss = 42 dB – 2.8 dB – 17.4 dB + 16 dB = 37.8 dB

Allowed L2 section loss = 37.8 dB

Using 0.58 dB/km for the loss per unit length in the L2 optical fiber link, we next calculate the length of L2,

Length of L2 = (37.8 dB/0.58 dB/km) = 65.17 km

[NOTE: Many in the class did poorly on Problem 7 of Homework 3, so I decided to put a similar problem on this homework set for added practice. This category of problem is very important in designing optical fiber links. You need to be able to do this category of problem in general.]

Problem 6 Longitudinal Modes in FP Cavity (20 points)

We showed in class we could write the frequency spacing $\Delta f$ of adjacent longitudinal modes (i.e., standing waves) in a Fabry-Pérot cavity as

$$\Delta f = \frac{c}{2Ln} \quad \text{Equation (6.14) in Senior}$$

where $L$ is the cavity length, $c$ is the speed of light, and $n$ is the index of refraction for the material and $f$ is frequency.

(a) Derive the expression for the wavelength spacing $\Delta \lambda$ of adjacent longitudinal modes (and state your assumptions and showing your steps in the derivation thereof).
Solution: (See page 305, equation (6.16) in Senior, 3rd ed., for his solution)

At resonance, \(2 \left( \frac{2\pi n}{\lambda_m} \right) L = 2\pi \cdot m \) (\(m\) integer)

Therefore, \(\frac{2Ln}{\lambda_m} = m\); and \(m = \frac{2Ln}{c} f_m\)

where \(\lambda_m \cdot f_m = \frac{c}{n}\)

\(m = \frac{2Ln}{c} f_m\) and \(m - 1 = \frac{2Ln}{c} f_{m-1}\)

\(m - (m - 1) = 1 = \frac{2Ln}{c} [f_m - f_{m-1}] = \frac{2Ln}{c} \Delta f\)

or \(\Delta f = \frac{c}{2Ln}\) and \(f = \frac{c}{\lambda}\)

Make use of \(\frac{\Delta f}{f} = \frac{\Delta \lambda}{\lambda}; \Delta \lambda = \frac{\lambda \Delta f}{f}\)

\(\Delta \lambda = \frac{\lambda \Delta f}{f} = \frac{\lambda \left( \frac{c}{2Ln} \right)}{f} = \frac{\lambda^2}{2Ln} \quad \Leftarrow\)

An alternate solution: What is changing? The integer number \(m\).

The mode number is \(m = \frac{Ln}{(\lambda / 2)} = \frac{2Ln}{\lambda} \) or \(\lambda = \frac{2Ln}{m}\)

Taking the derivative: \(\frac{d\lambda}{dm} = \frac{d}{dm} \left( \frac{2Ln}{m} \right) = -\frac{2Ln}{m^2}\)

Want \(\Delta \lambda\) for \(\Delta m = \pm 1; \Delta \lambda = \Delta m \left( \frac{2Ln}{m^2} \right) = \pm \frac{2Ln}{m^2}\)

But \(m = \frac{2Ln}{\lambda}\) and \(m^2 = \frac{4L^2 n^2}{\lambda^2}; \Delta \lambda = \frac{2Ln}{m^2} = \frac{\lambda^2}{2Ln} = \frac{\lambda^2}{2Ln} \quad \Leftarrow\)
(b) In part (a) you assumed the index of refraction to be constant. Now assume that the index of refraction is a function of wavelength \( n(\lambda) \). Derive an expression for the wavelength spacing \( \Delta \lambda \) of adjacent longitudinal modes with a dependent \( n(\lambda) \).

**Solution: (No one lost points on part (b).)**

Here is one possible approach to this problem:

Let's express the index of refraction in a Taylor's series of \( n(\lambda) \),

\[
n(\lambda) = n(\bar{\lambda}) + \left( \frac{dn(\lambda)}{d\lambda} \right) (\lambda - \bar{\lambda}) + \frac{1}{2!} \left( \frac{d^2 n(\lambda)}{d\lambda^2} \right) (\lambda - \bar{\lambda})^2 + \ldots
\]

where \( \bar{\lambda} \) is the average wavelength centered on the gain window of the laser's operational band.

Taking the first two terms, and substituting into the expression you found in part (a) for \( n \), gives an approximation for \( \Delta \lambda \),

\[
\Delta \lambda = \frac{\lambda^2}{2L \left[ n(\bar{\lambda}) + \left( \frac{dn(\lambda)}{d\lambda} \right) (\lambda - \bar{\lambda}) \right]} \quad \Leftarrow
\]

This is one of several ways part (b) can be approached.

Note that when \( \frac{dn(\lambda)}{d\lambda} \) is zero, we obtain,

\[
\Delta \lambda = \frac{\lambda^2}{2L \left[ n(\bar{\lambda}) \right]}, \quad \text{as would be expected.}
\]

**Problem 7  Bandgap in a Semiconductor (10 points)**

Explain the difference between direct bandgap and indirect bandgap semiconductors. What does it mean for semiconductor laser operation? In other words, which would be best for a diode laser?

**ANSWER:**

The recombination process can be explained with the material’s band structure. Generally, there are two kinds of band structure, direct band gap and indirect bandgap. Direct band gap means that in the E-k diagram, electrons at the minimum of the conduction band have the same crystal momentum as electrons...
at the maximum of the valence band; for an indirect band gap, the electrons do not have the same crystal momentum as the holes at the top of the valence band, as indicated in the diagram below. The recombination of an electron near the bottom of the conduction band with a hole near the top of the valence band requires the exchange of both energy and crystal momentum. For indirect band gap recombination, the energy may be carried off by a photon, but one or more phonons are also required to conserve crystal momentum. This multiparticle interaction has a lower probability of happening and so the recombination efficiency in the indirect band gap material is lower than in the direct band gap material. We desire direct bandgap semiconductors for making semiconductor lasers and LEDs.

Diagram for visualizing the above discussion:

![Diagram](https://www.laserdiodesource.com/laser-diode-technical-overview-one)

*Figure 5: Recombination [5]*