EE 443 Homework #8 Solutions
(Fall 2020 – Due November 19, 2020)
Print out homework and do work on the printed pages.


Problem 1 SNR for BER of 10^{-5} (20 points)

Using the Gaussian approximation determine the required signal-to-noise ratios (both optical and electrical) to maintain a BER = 10^{-5} on a baseband binary digital optical signal fiber link. Assume the decision threshold is set at the midpoint between the 1 and the 0 states. Use the fact that 2 \times 10^{-5} = \text{erfc}(3.01572). Reference: Section 12.6.3 (pages 715 to 725) in Senior, 3rd ed.

Solution:

We start with

\[ P(e) = \frac{1}{2} \text{erfc} \left( \frac{\sqrt{S/N}}{2\sqrt{2}} \right) = 10^{-5}; \quad \text{erfc} \left( \frac{\sqrt{S/N}}{2\sqrt{2}} \right) = 2 \times 10^{-5} \]

Thus,

\[ \frac{\sqrt{S/N}}{2\sqrt{2}} = 3.01573; \quad \frac{S}{N} = 2 \left( 3.01573 \cdot \sqrt{2} \right) = 8.6474 \]

Now we make use of the figure in Lecture 20, slide #41 (Figure 12.38(b) on page 719 of Senior) to find the optical SNR in terms of the peak signal current and rms noise current, that is [using eq. (12.23)]

\[ \frac{i_{\text{sig}}^2}{i_{N}^2} = \left( \frac{S}{N} \right)^{1/2} = 8.647 \quad \text{or} \quad 9.72369 \text{ dB} \]

The electrical SNR is given by eq. (12.24),

\[ \frac{i_{\text{sig}}^2}{i_{N}^2} = \left( \frac{S}{N} \right) = 74.778 \quad \text{or} \quad 18.74 \text{ dB} \]

For a program to evaluate erf(x) and erfc(x) values go to:
https://keisan.casio.com/exec/system/1180573448
Problem 2  RC Rise Time and Bandwidth (20 points)

You have an RC circuit as shown below with an applied unit step function $u(t)$. Its response is the well-known rising exponential function as shown in the lower right hand plot below.

![RC Circuit Diagram]

$$V_0(t) = V(1 - \exp(-t / RC))$$

(a) We define the rise time of the output to be the time to go from 10% of the final voltage value to 90% of the final voltage value. Using this definition of rise time show (by derivation) that

$$\tau_{\text{rise}} = 2.2RC$$

Solution:

\[
\frac{V_0(t)}{V} = 0.1 = (1 - \exp(-t_{0.1} / RC) \quad \Rightarrow \quad 0.9 = \exp(-t_{0.1} / RC)
\]

\[
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\]

So, $t_{0.1} = 0.1054 \cdot RC$ and $t_{0.9} = 2.3026 \cdot RC$

$$\tau_{\text{rise}} = t_{0.9} - (t_{0.1}) = (2.3026 - 0.1054)RC = 2.1972\cdot RC \quad \text{or} \quad 2.2RC$$
(b) Given that the bandwidth $B$ of the $RC$ circuit is the reciprocal of $(2\pi RC)$, show the rise time can be expressed as

$$\tau_{\text{rise}} = \frac{0.35}{B}$$

**Solution:**

Given that $B = \frac{1}{2\pi RC}$, and

$$\tau_{\text{rise}} = 2.2RC$$

$$B = \frac{2.2}{2\pi\tau_{\text{rise}}} \Rightarrow \tau_{\text{rise}} = \frac{0.35}{B}$$

**Problem 3 Comparing Two Optical Fiber Systems (40 points)**

On slide 16 (of Lecture 16) we showed a link-loss budget graph of an optical fiber link. Make a graphical comparison, and a spreadsheet table (as shown on slide 17 of Lecture 16) comparison of the two optical fiber systems defined below. Determine the maximum attenuation-limited transmission distance of both systems operating at 100 Mbps. There are no splices in the optical fiber (it is a single fiber strand).

**System 1**

Operates at 850 nm wavelength
GaAlAs laser diode: 0 dBm coupled-to-fiber output power
Silicon avalanche photodiode: -50 dBm sensitivity
Graded-index fiber: 4.0 dB/km attenuation at 850 nm
Connector loss: 1 dB/connector
Allow for 6 dB operating power margin.

**System 2**

Operates at 1300 nm wavelength
InGaAsP LED diode: -13 dBm coupled-to-fiber output power
InGaAs PIN photodiode: -38 dBm sensitivity
Graded-index fiber: 1.5 dB/km attenuation at 1300 nm
Connector loss: 1 dB/connector
Allow for 6 dB operating power margin.

Remember to include both the computation table used and the graphical drawing. Clearly state any assumptions you made in the calculations.
Solution:
Assume two connectors for a total loss of 2 dB and with 6 dB of margin. That is 8 dB subtracted from the 0 dBm input power. Then with a sensitivity of −50 dBm, we can allow 42 dB of fiber attenuation. A 4.0 dB/km, we find 42 dB/4.0 dB/km = 10.5 km in distance.

<table>
<thead>
<tr>
<th>Component or Process</th>
<th>Output/Sensitivity/loss</th>
<th>Power margin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Source power output</td>
<td>0 dBm</td>
<td></td>
</tr>
<tr>
<td>Photodetector sensitivity</td>
<td>-50 dBm</td>
<td></td>
</tr>
<tr>
<td>Allowed loss in link</td>
<td></td>
<td>50 dB</td>
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<tr>
<td>Connector loss (two connectors)</td>
<td>2 dB</td>
<td>48 dB</td>
</tr>
<tr>
<td>Link margin (6 dB)</td>
<td>6 dB</td>
<td>42 dB</td>
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<tr>
<td>Fiber loss is 4.0 dB/km</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
42 \text{ dB}/4.0 \text{ dB/km} = 10.5 \text{ km}
\]
Solution:
Assume two connectors for a total loss of 2 dB and with 6 dB of margin. That is 8 dB subtracted from the -13 dBm input power. Then with a sensitivity of \(-38 \text{ dBm}\), we can allow for 17 dB of fiber attenuation. At 1.5 dB/km, we find 17 dB/1.5 dB/km = 11.33 km distance.

<table>
<thead>
<tr>
<th>Component or Process</th>
<th>Output/Sensitivity/loss</th>
<th>Power margin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Source power output</td>
<td>-13 dBm</td>
<td></td>
</tr>
<tr>
<td>Photodetector sensitivity</td>
<td>-38 dBm</td>
<td></td>
</tr>
<tr>
<td>Allowed loss in link</td>
<td>25 dB</td>
<td></td>
</tr>
<tr>
<td>Connector loss (two connectors)</td>
<td>2 dB</td>
<td>23 dB</td>
</tr>
<tr>
<td>Link margin (6 dB)</td>
<td>6 dB</td>
<td>17 dB</td>
</tr>
<tr>
<td>Fiber loss is 1.5 dB/km</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>17 dB/1.5 dB/km = 11.33 km</td>
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</tbody>
</table>

Problem 4 Energy per Bit to Noise Ratio (20 points)

In digital communication or data transmission, \(E_b/N_0\) (energy per bit to noise power spectral density ratio) is a normalized signal-to-noise ratio (SNR) measure, also known
as the "SNR per bit". It is especially useful when comparing the bit error rate (BER) performance of different digital modulation schemes without taking bandwidth into account.

As the description implies, $E_b$ is the signal energy associated with each data bit; it is equal to the signal power divided by the bit rate. If signal power is in watts and bit rate is in bits per second, $E_b$ is in units of joules (watt-seconds). $N_0$ is the noise spectral density, that is, the noise power in a 1 Hz bandwidth, measured in watts per hertz or joules.

These are the same units as $E_b$ so the ratio $E_b/N_0$ is dimensionless; it is sometimes expressed in decibels. $E_b/N_0$ directly indicates the power efficiency of the system without regard to modulation type, error correction coding or signal bandwidth.

(a) If $R_b$ is the bit rate and $S$ is the signal power of a binary signal, prove that $E_b/N_0$ can be written as

$$\frac{E_b}{N_0} = \frac{S}{kT R_b}$$

where $k$ is Boltzmann’s constant and $T$ is temperature.

**Solution:**
The energy per bit $Eb$ is equal to the signal power $S$ divided by the bit rate $Rb$, thus, $E_b = \frac{S}{R_b}$.

The thermal noise power per hertz is $kT$, thus we write,

$$\frac{E_b}{N_0} = \frac{(C / R_b)}{kT} = \frac{S}{kT R_b} \Leftarrow$$

(b) Recall that the thermal noise power per hertz $N_0 = kT$ (W/Hz) and the noise power over a bandwidth of $B_T$ is $N = kTB_T$. Parameter $k$ is Boltzmann’s constant and $T$ is temperature. Find the relationship between the signal-to-noise ratio ($S/N$) and $E_b/N_0$.

**Solution:**
By definition, the total noise power $N$ is $kTBT$ where BT is the bandwidth of the channel and therefore,

$$\frac{S}{N} = \frac{S}{N_0 B_T} = \frac{E_b R_b}{N_0 B_T} = \left( \frac{E_b}{N_0} \right) \frac{R_b}{B_T} \Leftarrow$$

(c) Shannon’s theorem states that the maximum channel capacity $C$ (i.e., bit rate) is

$$C = B_T \times \log_2 \left( 1 + \frac{S}{N} \right)$$
Note: \( C = R_b \) = bit rate (bits/second). Find the signal-to-noise ratio \((S/N)\) using Shannon’s theorem to express it as a ratio of \(C/B_T\).

**Solution:**

\[
C = B_T \cdot \log_2 \left( 1 + \frac{S}{N} \right) \quad \text{so} \quad \frac{C}{B_T} = \log_2 \left( 1 + \frac{S}{N} \right)
\]

\[
2^{(C/B_T)} = \left( 1 + \frac{S}{N} \right) \quad \text{and} \quad \frac{S}{N} = \left( 2^{(C/B_T)} - 1 \right) \quad \Leftarrow
\]

Also,

\[
\frac{E_b}{N_0} = \frac{B_T}{C} \left( 2^{(C/B_T)} - 1 \right)
\]

(d) What is the value of \( E_b/N_0 \) if the spectral density is equal to 6 bps/Hz?

**Solution:**

\[
\frac{E_b}{N_0} = \frac{B_T}{C} \left( 2^{(C/B_T)} - 1 \right) = \frac{1}{6} \left( 2^6 - 1 \right) = \frac{(64 - 1)}{6} = 10.50 \quad \Leftarrow
\]