Problem 1  Non-concentric Misaligned Fibers  (20 points)

Consider the case where two optical fibers are being spliced together. Assume that both fiber ends are perfectly cleaved and there is no air space between the two ends of the fibers. However, they are not concentric, that is, they are misaligned. Both fibers have core radii $r = 10$ micrometers ($\mu$m), and the centers of the cores are offset by a distance $d = 12$ $\mu$m. This is illustrated in the cross-sectional diagram below which defines the symbols used. Assume refractive index of both cores to be $n = 1.50$.

The overlapping cross-sectional area establishes the fraction of optical power transferred from one fiber to the mating fiber. Obviously, the optical power is reduced by the misalignment and we want to know by how much it is reduced as a function of distance $d$. Be sure to use the symmetry from the core diameters being equal, that is $r_1 = r_2 = r$. This simplifies the derivation asked for below.

**Hint:**

Use the figure below as a guide to derive an expression for the overlap area of the two fibers (which will be twice the area labelled $A_2$). Angle $2\alpha$ is the sector of the circle that
overlaps with the other fiber. Knowing the angle $2\alpha$ allows the area $(A_1 + A_2)$ to calculated and then the triangular area $A_1$ can be subtracted to give area $A_2$.

**Derivation:**
We want to find $Area(A_2)$ because the area of fiber overlap will be twice $Area(A_2)$. Our plan is to find $Area(A_1+A_2)$, subtract $Area(A_1)$, then double for the total overlap area. We use the figure below to guide us on our derivation.

We start with finding an expression for $Area(A_1+A_2)$. This can be found by observing that the total cross-sectional area $A_{\text{max}}$ of the fiber is simply equal to

$$A_{\text{max}} = \pi r^2$$

and the area of the wedge of the circle containing areas $A_1$ and $A_2$ is the fraction

$$Area(A_1+A_2) = \left(\frac{2\alpha}{360}\right) A_{\text{max}} = \left(\frac{2\alpha}{360}\right) \pi r^2, \quad (\alpha \text{ in degrees})$$

Extra points (up to 10 points) are given for the full derivation showing all steps in the derivation.

(a) Derive an expression for the area of the shaded (overlapping) area. Show that the expression for the fiber-to-fiber areal overlap is given by

$$A = 2 \times \left[ r^2 \cdot \cos^{-1} \left( \frac{d}{2r} \right) - \frac{d}{2} \sqrt{r^2 - \left( \frac{d}{2} \right)^2} \right]$$

We can use the figure below to guide us on our derivation.
Next, we find the angle $\alpha$ from the radius $r$ and $d/2$. From trigonometry we find

$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{(d/2)}{r}; \quad \text{thus} \quad \alpha = \cos^{-1}\left(\frac{d}{2r}\right)$$

Assuming the angle $\alpha$ is in radians, we have

$$\text{Area}(A1+A2) = \left(\frac{2\alpha}{2\pi}\right)\left(\pi r^2\right) = r^2\cos^{-1}\left(\frac{d}{2r}\right)$$

Now we find Area(A1) using the Pythagorean Theorem, so the opposite side $\eta$ of the lower triangle is given by

$$r^2 = \left(\frac{d}{2}\right)^2 + \eta^2; \quad \text{thus} \quad \eta = \sqrt{r^2 - \left(\frac{d}{2}\right)^2}$$

$$\text{Area}(A1) = \frac{d}{2} \times \sqrt{r^2 - \left(\frac{d}{2}\right)^2}$$

Finally, $\text{Area}(A2) = \text{Area}(A1+A2) - \text{Area}(A1)$, thus

$$\text{Area}(A2) = r^2 \times \cos^{-1}\left(\frac{d}{2r}\right) - \frac{d}{2} \times \sqrt{r^2 - \left(\frac{d}{2}\right)^2}$$

But the total fiber overlap area $A$ is twice Area(A2), so the final answer is

$$A = 2 \times \left[ r^2 \times \cos^{-1}\left(\frac{d}{2r}\right) - \frac{d}{2} \times \sqrt{r^2 - \left(\frac{d}{2}\right)^2} \right]$$
(b) What fraction of the optical power is transmitted using the values for the fiber core radius \( r = 25 \, \mu m \) and misalignment separation of \( d = 10 \, \mu m \). State any assumptions you made. Compare the area you calculate to the maximum area when both fibers are perfectly aligned, namely area \( A_{\text{max}} = \pi r^2 \, (\mu m)^2 \), thereby giving the fraction of the optical signal transmitted.

**Solution:** Remember to use radians with the \( \cos^{-1}(x) \) term.

\[
A = 2 \times \left[ \frac{r^2 \cdot \cos^{-1} \left( \frac{d}{2r} \right) - d/2 \sqrt{r^2 - \left( \frac{d}{2} \right)^2}}{r^2} \right]
\]

\[
A = 2 \times \left[ 25^2 \cdot \cos^{-1} \left( 10 \right) - 5 \sqrt{(25)^2 - (5)^2} \right]
\]

\[
A = 2 \times \left[ 625 \cdot \cos^{-1} (0.20) - 5 \sqrt{625 - 25} \right] = 1466.3 \, \mu m^2 \quad \leftarrow
\]

For the fraction of the optical signal transmitted,

\[
\frac{A}{A_{\text{max}}} = \frac{1466.3 \, \mu m^2}{1963.5 \, \mu m^2} = 0.747 \quad \leftarrow
\]

This means that only 28.5% of the optical signal is transmitted at the splice (71.5% is lost).

**Problem 2 Two Aligned Fibers but with Airgap** (10 points)

Two optical fibers are spliced together but with an air gap, \( d_g = 3 \, \text{micrometers} \); however, they are centerline aligned and the cleaved ends of each fiber are smooth and perpendicular. The cores of the two fibers both have a refractive index \( n_1 = 1.50 \) and the diameter \( D \) of both cores is 50 micrometers. The magnitude of the partial reflection \( r \) of light transmitted through the air gap interface is calculated using the classical Fresnel formula for normal incidence which is (also, note that \( n = 1 \) for air):

\[
r = \left( \frac{n_1 - n}{n_1 + n} \right)^2
\]

This equation is for one fiber core to air interface and for an air gap between two fibers being mated there are two interfaces, so it must be applied twice. How much loss does
the 3 μm air gap introduce in the fiber splice? Express your answer in decibels (dB) where loss in dB is given by the Fresnel loss defined by

\[
\text{Fresnel Loss} = -10 \cdot \log_{10}(1 - r)
\]

**Solution:**

\[
r = \left( \frac{n_2 - n}{n_1 + n} \right)^2 = \left( \frac{1.5 - 1}{1.5 + 1} \right)^2 = \left( \frac{0.5}{2.5} \right)^2 = 0.04
\]

\[
\text{Loss} = -10 \cdot \log(1 - r),
\]

but there are two interfaces, so the loss is actually

\[
\text{Loss} = -10 \cdot \log(1 - 2r) = -10 \cdot \log(0.92) = 0.362 \text{ dB}
\]

**Problem 3 Two Aligned Fibers with Index Gel (10 points)**

Repeat the calculation from Problem 2 above, but now with a gel replacing the air gap. The effective index of refraction of the gel is \(n_{\text{gel}} = 1.46\). Note: This would be typical for a fiber splice.

(a) What is the loss (in dB) for the gel gilled splice of these two fibers?

**Solution:**

\[
r = \left( \frac{n_2 - n}{n_1 + n} \right)^2 = \left( \frac{1.5 - 1.46}{1.5 + 1.46} \right)^2 = \left( \frac{0.04}{2.96} \right)^2 = (0.01351)^2 = 0.0001826
\]

\[
\text{Loss} = -10 \cdot \log(1 - 2r) = -10 \cdot \log(0.999635) = 0.001587 \text{ dB}
\]

(b) What fractional improvement does the gel give over the air gap case in part (a) above?

**Solution:**

The loss with the air gap is \(10^{-0.362/10} = 0.9200\) or about an 8.00% loss

The loss with the gel included is \(10^{-0.001587/10} = 0.99963\) or about 0.037% loss

The ratio is an improvement of approximately 220 times.

**Problem 4 Lithium Niobate Modulator (10 points)**
Given a lithium niobate strip waveguide as shown in the figure below, determine the voltage required to give a phase shift of $\pi$ radians. The wavelength $\lambda$ of operation is 1310 nm, the length $L$ of the modulator = 1.5 cm, the spacing $d$ between the two electrodes is 25 micrometers (\(\mu\)m), the refractive index of lithium niobate is $n = 2.1$ and the electro-optic coefficient $r$ is $31 \times 10^{-12}$ meters/volt.

![Diagram of lithium niobate strip waveguide](image)

**Solution:**

When the phase change is $\pi$ radians, using eq. (11.12) we have

$$\delta \phi = \pi = \frac{\pi}{\lambda} n^3 r \frac{V \pi L}{d}$$

Hence, the voltage required to produce a phase change of $\pi$ radians is given by

$$V_\pi = \frac{\lambda d}{n^3 r L} = \frac{(1.31 \times 10^{-6})(25 \times 10^{-6})}{(2.1)^3 \times (31 \times 10^{-12}) \times (1.5 \times 10^{-2})}$$

$$V_\pi = 7.605 \text{ volts}$$

**Problem 5 2x2 biconical tapered coupler  (15 points)**

In this problem we have a 2 x 2 single-mode biconical tapered coupler with a 40/60 splitting ratio where the insertion losses are 2.7 dB for the 60% channel and 4.7 dB for the 40% channel. Refer to Section 5.6.1 starting on page 259.

(a) Given an input power $P_0$ of 200 microwatts (\(\mu\)W), find the output powers of both the 40% channel and the 60% channel:
Solution:

\[
10 \log_{10} \left( \frac{200 \times 10^{-6}}{P_{60\%}} \right) = 2.7 \text{ dB}
\]

\[
P_{60\%} = \left( 200 \times 10^{-6} \right) / 10(2.7/10) = 107.4 \times 10^{-6} \text{ W}=107.4 \mu\text{W}
\]

\[
10 \log_{10} \left( \frac{200 \times 10^{-6}}{P_{40\%}} \right) = 4.7 \text{ dB}
\]

\[
P_{40\%} = \left( 200 \times 10^{-6} \right) / 10(4.7/10) = 67.8 \times 10^{-6} \text{ W}=67.8 \mu\text{W}
\]

(b) Find the excess loss of the coupler.

\[
\text{Excess loss} = 10 \log_{10} \left( \frac{200 \times 10^{-6}}{107.4 \times 10^{-6} + 67.8 \times 10^{-6}} \right) = 0.575 \text{ dB}
\]

Problem 6 Star Coupler (20 points)

Star couplers are covered in Senior, Section 5.6.2, pages 264 to 268. An 8 x 8 star coupler is shown in the figure below from Lecture 19. A key feature of a star coupler is that the optical power from any one of the input fibers is uniformly distributed among all of the output fibers.

![Fiber fused biconical taper 8 x 8 port star coupler](image)

**Figure 2.37** Fiber fused biconical taper 8 x 8 port star coupler

For a single input port and multiple output ports where \( j = 1, N \), then the excess loss is given by:

\[
\text{Excess loss (star coupler)} = 10 \log_{10} \left( \frac{P_j}{\sum_{j=1}^{N} P_j} \right) \text{ (dB)}
\]
Suppose 1 mW of power is directed into one of the eight input fibers. The output power from each of the output fibers is measured to be 0.088 mW (= 88 μW).

(a) Calculate the splitting loss of this star coupler. The splitting loss is the expected loss in an ideal star coupler with one input and N outputs (and N = 8 for this star coupler).

Solution:

The splitting loss can be written as

\[
\text{Splitting loss} = 10 \cdot \log_{10}(N) = 10 \cdot \log_{10}(8) = 9.03 \text{ dB}
\]

(b) What is the excess loss for this star coupler?

Solution:

The excess loss is calculated from

\[
\text{Excess loss} = 10 \cdot \log_{10} \left( \sum_{j=1}^{8} \frac{P_i}{P_j} \right)
\]

\[
= 10 \cdot \log_{10} \left( \frac{1000 \text{ μW}}{8 \times 88 \text{ μW}} \right) = 10 \cdot \log_{10} (1.4205) = 1.524 \text{ dB}
\]

(c) What is the total loss for this star coupler?

Solution:

The total loss is the splitting loss plus the excess loss.

\[
\text{Total loss} = 9.03 \text{ dB} + 1.52 \text{ dB} = 10.55 \text{ dB}
\]

(d) What is the insertion loss for this star coupler (remember the insertion loss is the relationship of the output power on one output fiber and the total input power)?
Solution: The insertion loss is the total input power into the input fiber divided by the single output fiber’s power.

\[
\text{Insertion loss} = 10 \cdot \log_{10} \left( \frac{1000}{88} \right) = 10 \cdot \log_{10} (11.36) = 10.55 \text{ dB }
\]

Problem 7 Design a two-input-to-nine-output splitter (15 points)

Using the basic building block of a 3-dB coupler as drawn below, design a two-input-to-nine-output splitter. The building block is

FIGURE 12.5 If the length of the coupler is chosen appropriately, the incident power can be split equally between the two output fibers. Such a coupler is called a 3-dB coupler.

https://www.globalspec.com/reference/13962/160210/chapter-12-4-1-fiber-optic-components-fiber-optic-couplers

Solution:
One possible solution is shown below: