Useful Mathematical Relations for ES 442

Version 1.1

Euler’s formula: \[ \exp[\pm j\theta] = \cos(\theta) \pm j \sin(\theta) \]

Magnitude & phase:
\[ |\exp[\pm j\theta]| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1 \]
\[ \angle \exp[\pm j\theta] = \tan^{-1}\left( \pm \frac{\sin(\theta)}{\cos(\theta)} \right) = \tan^{-1}(\pm \tan(\theta)) = \pm \theta \]

Properties:
\[ \sin(\theta) = \frac{1}{2j} (\exp[j\theta] - \exp[-j\theta]) \]
\[ \cos(\theta) = \frac{1}{2} (\exp[j\theta] + \exp[-j\theta]) \]

Other:
\[ e^{\pm j(\pi/2)} = \pm j \quad \text{and} \quad e^{\pm jn\pi} = \begin{cases} 1 & \text{for } n \text{ even} \\ -1 & \text{for } n \text{ odd} \end{cases} \]
\[ a + jb = re^{i\theta} \quad \text{where} \quad r = \sqrt{a^2 + b^2}, \quad \theta = \tan^{-1}\left(\frac{b}{a}\right) \]
Sinusoidal Waveforms in Complex Plane

Only three parameters are needed to specify a phasor: $A$, $f$ and $\phi$.

$$A \cos(\omega t + \phi) = \text{Re}[A \, e^{i(\omega t + \phi)}]$$
$$A \sin(\omega t + \phi) = \text{Im}[A \, e^{i(\omega t + \phi)}]$$

(positive direction)

$\omega = 2\pi f$

$A \cos(\omega t + \phi) = [A \cos(\phi)]\cos(\omega t) - [A \sin(\phi)]\sin(\omega t)$
Phasors

\[ A \cdot \cos(2\pi f_0 t + \phi) = \frac{A}{2} e^{j\phi} e^{j2\pi f_0 t} + \frac{A}{2} e^{-j\phi} e^{-j2\pi f_0 t} \]

\[ \omega = 2\pi f \]

Amplitude

Phase

Conjugate term is needed to form a real function of time.
Trignometric Identities

\[
\exp[\pm j\theta] = \cos(\theta) \pm j\sin(\theta)
\]

\[
\sin(\theta) = \frac{1}{2j}(\exp[j\theta] - \exp[-j\theta])
\]

\[
\cos(\theta) = \frac{1}{2}(\exp[j\theta] + \exp[-j\theta])
\]

\[
\cos\left(\theta \pm \frac{\pi}{2}\right) = \mp \sin(\theta) \quad \text{and} \quad \sin\left(\theta \pm \frac{\pi}{2}\right) = \pm \cos(\theta)
\]

\[
2\sin(\theta) \cdot \cos(\theta) = \sin(2\theta)
\]

\[
\sin^2(\theta) + \cos^2(\theta) = 1
\]

\[
\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)
\]

\[
\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))
\]

\[
\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))
\]

\[
\sin(\theta \pm \phi) = \sin(\theta) \cdot \cos(\phi) \pm \cos(\theta) \cdot \sin(\phi)
\]

\[
\cos(\theta \pm \phi) = \cos(\theta) \cdot \cos(\phi) \mp \sin(\theta) \cdot \sin(\phi)
\]
Trigonometric Identities (continued)

\[
\sin(\theta) \cdot \sin(\phi) = \frac{1}{2} [\cos(\theta - \phi) - \cos(\theta + \phi)] \\
\cos(\theta) \cdot \cos(\phi) = \frac{1}{2} [\cos(\theta - \phi) + \cos(\theta + \phi)] \\
\sin(\theta) \cdot \cos(\phi) = \frac{1}{2} [\sin(\theta - \phi) + \sin(\theta + \phi)] \\
\tan(\theta \pm \phi) = \frac{\tan(\theta) \pm \tan(\phi)}{1 \mp \tan(\theta) \cdot \tan(\phi)} \\
\]

\[
a \cos(\theta) + b \sin(\theta) = \sqrt{a^2 + b^2} \cdot \cos \left[ \theta + \tan^{-1} \left(\frac{-b}{a}\right) \right] \\
\]

\[
\cos^3(\theta) = \frac{1}{4} (3 \cos(\theta) + \cos(3\theta)) \\
\sin^3(\theta) = \frac{1}{4} (3 \sin(\theta) - \sin(3\theta))
\]
Power Series

Taylor series: \( f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots. \)

where \( f^{(n)}(a) = \frac{d^n f(x)}{dx^n} \bigg|_{x=a} \)

MacLaurin series: \( f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots. \)

where \( f^{(n)}(0) = \frac{d^n f(x)}{dx^n} \bigg|_{x=0} \)

\( e^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots. \)

\( \sin(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots. \quad \cos(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots. \)

\( \tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots. \quad \tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots; \quad |x| < 1 \)

\( \sin c(x) = 1 - \frac{1}{3!}(\pi x)^2 + \frac{1}{5!}(\pi x)^4 - \frac{1}{7!}(\pi x)^6 + \cdots. \)
Power Series (continued)

\[
\log(x) = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \cdots + \frac{1}{n}\left(\frac{x-1}{x}\right)^n + \cdots; \text{ for } x > \frac{1}{2}
\]

\[
\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.
\]

\[
a^x = 1 + x \cdot \log_e(a) + \frac{\left[x \cdot \log_e(a)\right]^2}{2!} + \cdots + \frac{\left[x \cdot \log_e(a)\right]^n}{n!} + \cdots.
\]

\[
\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots \text{ for } x^2 < 1
\]

\[
\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots; \text{ for } |x| < 1
\]

\[
(1+x)^n = 1 + n \cdot x + \frac{n(n-1)}{2!} \cdot x^2 + \frac{n(n-1)(n-2)}{3!} \cdot x^3 + \cdots + \binom{n}{k} x^k + n^n
\]

where the binomial coefficient is \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \)

\[
(1+x)^n \approx 1 + n \quad \text{for } |x| << 1
\]
Some Indefinite Integrals

\[ \int udv = uv - \int vdu \]
\[ \int \sin(ax)dx = -\frac{1}{a} \cos(ax) \quad \text{and} \quad \int \cos(ax)dx = \frac{1}{a} \sin(ax) \]
\[ \int \sin^2(ax)dx = \frac{x}{2} - \frac{\sin(2ax)}{4a} \quad \text{and} \quad \int \cos^2(ax)dx = \frac{x}{2} + \frac{\sin(2ax)}{4a} \]
\[ \int x \cdot \sin(ax)dx = \frac{1}{a^2} (\sin(ax) - ax \cdot \cos(ax)) \]
\[ \int x \cdot \cos(ax)dx = \frac{1}{a^2} (\cos(ax) + ax \cdot \sin(ax)) \]
\[ \int \sin(ax) \cdot \sin(bx)dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} ; \quad a^2 \neq b^2 \]
\[ \int \sin(ax) \cdot \cos(bx)dx = -\left[ \frac{\cos(a-b)x}{2(a-b)} + \frac{\cos(a+b)x}{2(a+b)} \right] ; \quad a^2 \neq b^2 \]
\[ \int \cos(ax) \cdot \cos(bx)dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} ; \quad a^2 \neq b^2 \]
Some Indefinite Integrals (continued)

\[ \int e^{ax} \, dx = \frac{1}{a} e^{ax} \]

\[ \int xe^{ax} \, dx = \frac{e^{ax}}{a^2} (ax - 1) \]

\[ \int x^2 e^{ax} \, dx = \frac{e^{ax}}{a^3} (a^2x^2 - 2ax + 2) \]

\[ \int e^{ax} \cdot \sin(bx) \, dx = \frac{e^{ax}}{a^2 + b^2} \left[ a \sin(bx) - b \cos(bx) \right] \]

\[ \int e^{ax} \cdot \cos(bx) \, dx = \frac{e^{ax}}{a^2 + b^2} \left[ a \cos(bx) + b \sin(bx) \right] \]

\[ \int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) \]

\[ \int \frac{x}{x^2 + a^2} \, dx = \frac{1}{2} \ln(x^2 + a^2) \]
Selected Functions

Rectangular
\[ \Pi \left( \frac{t}{\tau} \right) = \text{rect} \left( \frac{t}{\tau} \right) = \begin{cases} 1 & \text{if } |t| \leq \frac{\tau}{2} \\ 0 & \text{if } |t| > \frac{\tau}{2} \end{cases} \]

Triangular
\[ \Delta \left( \frac{t}{\tau} \right) = \begin{cases} 1 - \frac{t}{\tau} & \text{for } |t| \leq \tau \\ 0 & \text{for } |t| > \tau \end{cases} \]

Sinc
\[ \text{sinc}(x) = \text{Sa}(x) = \frac{\sin(x)}{x} \]

Unit step
\[ u(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} \]

Signum
\[ \text{sgn}(t) = \begin{cases} 1 & \text{for } t > 0 \\ -1 & \text{for } t < 0 \end{cases} \]

Impulse
\[ \delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases} \]