6. MAXWELL’S EQUATIONS IN TIME-VARYING FIELDS
Applets

- [http://www.oerrecommender.org/visits/119103](http://www.oerrecommender.org/visits/119103)
- [http://webphysics.davidson.edu/physlet_resources/ bu_semester2/](http://webphysics.davidson.edu/physlet_resources/ bu_semester2/)
So far....

- **Static Electromagnetic ....**
  - No change in time (static)

- We now look at cases where currents and charges vary in time → H & E fields change accordingly
  - Examples: light, x-rays, infrared waves, gamma rays, radio waves, etc

- We refer to these waves as **time-varying** electromagnetic waves
  - A set of new equations are required!
In this chapter, we will examine Faraday’s and Ampère’s laws.
A little History

- **Oersted** demonstrated the relation between electricity and magnetism
  - Current impacts (excerpts force on) a compass needle
    - Fm is due to the magnetic field
    - When current in Z then needle moves to Phi direction


Induced magnetic field can influence the direction of the compass needle. When we connect the circuit, the conducting wire wrapped around the compass is energized creating a **magnetic field** that counteracts the effects of the Earth's magnetic field and changes the direction of the compass needle.
Faraday (in London) hypothesized that magnetic field should induce current!

Henry in Albany independently trying to prove this!

They showed that magnetic fields can produce electric current

The key to this induction process is CHANGE


Another applet: http://phet.colorado.edu/sims/faradays-law/faradays-law_en.html
Faraday’s Law

Electromotive force (voltage) induced by time-varying magnetic flux:

\[ V_{\text{emf}} = -N \frac{d\Phi}{dt} = -N \frac{d}{dt} \oint_{S} \mathbf{B} \cdot d\mathbf{s} \quad (V) \]

We can generate the current through the loop by moving the loop or changing direction of current.

Magnetic fields can produce an electric current in a closed loop,

When the meter detects current \(\rightarrow\) voltage has been induced \(\rightarrow\) electromotive force has been created \(\rightarrow\) this process is called emf induction.

http://phet.colorado.edu/sims/faradays-law/faradays-law_en.html

http://phet.colorado.edu/sims/faradays-law/faradays-law_en.html
Three types of EMF

1. A time-varying magnetic field linking a stationary loop; the induced emf is then called the *transformer emf*, $V_{\text{emf}}^{\text{tr}}$.

2. A moving loop with a time-varying surface area (relative to the normal component of $\mathbf{B}$) in a static field $\mathbf{B}$; the induced emf is then called the *motional emf*, $V_{\text{emf}}^{\text{m}}$.

3. A moving loop in a time-varying field $\mathbf{B}$.

The total emf is given by

$$V_{\text{emf}} = V_{\text{emf}}^{\text{tr}} + V_{\text{emf}}^{\text{m}}.$$
Stationary Loop in Time-Varying $B$

$V_{\text{emf}}^{\text{tr}} = -N \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$ (transformer emf),

$I = \frac{V_{\text{emf}}^{\text{tr}}}{R + R_i}$.  \hspace{1cm} (6.9)

For good conductors, $R_i$ usually is very small, and it may be ignored in comparison with practical values of $R$.

**Figure 6-2:** (a) Stationary circular loop in a changing magnetic field $\mathbf{B}(t)$, and (b) its equivalent circuit.
Stationary Loop in Time-Varying $\mathbf{B}$

- Assuming $S$ is stationary and $\mathbf{B}$ is varying:
  Transformer EMF
  
  $V_{\text{emf}}^{\text{tr}} = -N \int_{S} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$ (transformer emf),

- Two types of $\mathbf{B}$ fields are generated
  - Changing $\mathbf{B}(t)$
  - Induced $\mathbf{B}$ ($\mathbf{B}_{\text{ind}}$)

- Applet:
  
Lenz’s Law

- Increasing $B(t) \rightarrow$ change of magnetic flux $\rightarrow I(t)$ generates;
  - $I(t) \rightarrow$ Bind
  - Bind with be opposite of $B(t)$ change
  - Direction of Bind $\rightarrow$ Direction of $I(t)$
Lenz’s Law

- Increasing B(t) $\rightarrow$ change of magnetic flux $\rightarrow$ I generates;
  - I(t) $\rightarrow$ Bind
  - Bind with be opposite of B(t) change
  - Direction of Bind $\rightarrow$ Direction of I(t)
  - If I(t) clockwise $\rightarrow$ I moving from + to –
    - V2 > V1 $\rightarrow$ emf is negative

Figure 6-2: (a) Stationary circular loop in a changing magnetic field $B(t)$, and (b) its equivalent circuit.
Faraday’s Law

A magnetic field induces an E field whose CURL is equal to the negative of the time derivative of B

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]

\[ \oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot ds \]

\[ \Phi = \int \mathbf{B} \cdot ds \]

Faraday’s Law

\[ V_{\text{emf}} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \int_C \mathbf{B} \cdot ds = V_{\text{emf}}^{\text{tr}} + V_{\text{emf}}^{\text{m}} \]

Transformer

\[ V_{\text{emf}}^{\text{tr}} = -N \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot ds \quad \text{(N loops)} \]

Motional

\[ V_{\text{emf}}^{\text{m}} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \]
An inductor is formed by winding \( N \) turns of a thin conducting wire into a circular loop of radius \( a \). The inductor loop is in the \( x-y \) plane with its center at the origin, and connected to a resistor \( R \), as shown in Fig. 6-3. In the presence of a magnetic field \( \mathbf{B} = B_0 (\hat{y}2 + \hat{z}3) \sin \omega t \), where \( \omega \) is the angular frequency, find

(a) the magnetic flux linking a single turn of the inductor,
(b) the transformer emf, given that \( N = 10, \ B_0 = 0.2 \) T, \( a = 10 \) cm, and \( \omega = 10^3 \) rad/s,
(c) the polarity of \( V_{\text{emf}} \) at \( t = 0 \), and
(d) the induced current in the circuit for \( R = 1 \) k\( \Omega \) (assume the wire resistance to be much smaller than \( R \)).
Finding the Magnetic Flux:

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s}$$

Find ds!

Find Vtr:

Find Vrt!

Find Polarity of Vtr at t=0

What is V1 – V2 based on the Given polarity?

Note B0
Finding the Magnetic Flux:

\[ \Phi = \int_S \mathbf{B} \cdot d\mathbf{s} \]

\[ = \int_S [B_0(\hat{\mathbf{y}} 2 + \hat{\mathbf{z}} 3) \sin \omega t] \cdot \hat{\mathbf{z}} \, ds \]

\[ = 3\pi a^2 B_0 \sin \omega t. \]

Find Vtr:

\[ V_{\text{emf}}^{\text{tr}} = -N \frac{d\Phi}{dt} \]

\[ = -\frac{d}{dt} (3\pi N a^2 B_0 \sin \omega t) \]

\[ = -3\pi N \omega a^2 B_0 \cos \omega t. \]

Find Polarity of Vtr at t=0

\[ V_{\text{emf}}^{\text{tr}} = V_1 - V_2 \]

\[ = -188.5 \quad \text{(V)}. \]

For \( N = 10, \ a = 0.1 \text{ m}, \ \omega = 10^3 \text{ rad/s}, \ \text{and} \ B_0 = 0.2 \text{ T}, \)

\[ V_{\text{emf}}^{\text{tr}} = -188.5 \cos 10^3 t \quad \text{(V)}. \]
Ideal Transformer

\[ V_1 = -N_1 \frac{d\Phi}{dt}. \]

A similar relation holds true on the secondary side:

\[ V_2 = -N_2 \frac{d\Phi}{dt}. \]

Due to the primary coil

\[ \frac{V_1}{V_2} = \frac{N_1}{N_2} \]

\[ \frac{I_1}{I_2} = \frac{N_2}{N_1} \]

\[ R_{in} = \frac{V_1}{I_1} \]

\[ R_{in} = \frac{V_2}{I_2} \left( \frac{N_1}{N_2} \right)^2 = \left( \frac{N_1}{N_2} \right)^2 R_L. \] (6.20)

When the load is an impedance \( Z_L \) and \( V_1 \) is a sinusoidal source, the phasor-domain equivalent of Eq. (6.20) is

\[ Z_{in} = \left( \frac{N_1}{N_2} \right)^2 Z_L. \] (6.21)

Primary and sec. coils
Separated by the magnetic core
(permittivity is infinity)
Magnetic flux is confined in the core
Motional EMF

- In the existence of constant (static) magnetic field the wire is moving
  - $F_m$ is generated in charges
  - Thus, $E_m = \frac{F_m}{q} = UXB$
  - $E_m$ is motional emf

- [http://webphysics.davidson.edu/physlet_resources/bu_semester2/](http://webphysics.davidson.edu/physlet_resources/bu_semester2/) (motional EMF)
Motional EMF

Magnetic force on charge \( q \) moving with velocity \( u \) in a magnetic field \( B \):
\[
F_m = q(u \times B).
\]

This magnetic force is equivalent to the electrical force that would be exerted on the particle by the electric field \( E_m \) given by
\[
E_m = \frac{F_m}{q} = u \times B.
\]

This, in turn, induces a voltage difference between ends 1 and 2, with end 2 being at the higher potential. The induced voltage is
\[
V_{emf} = V_{12} = \int_0^1 E_m \cdot dl = \int_0^1 (u \times B) \cdot dl.
\]
Example: Sliding Bar

The length of the loop is related to \( u \) by \( x_0 = ut \). Hence

\[
V_{\text{emf}}^m = V_{12} = V_{43} = \int_3^4 \left( u \times B \right) \cdot dl
\]

\[
= \int_3^4 (\hat{x}u \times \hat{z}B_0x_0) \cdot \hat{y} \, dl = -uB_0x_0l.
\]

Note that \( B \) increases with \( x \)

\[
B = \hat{z}B_0x
\]

The length of the loop is related to \( u \) by \( x_0 = ut \). Hence

\[
V_{\text{emf}}^m = -B_0u^2lt \quad (V).
\]
EM Generator
**EM Motor/Generator Reciprocity**

**Motor:** Electrical to mechanical energy conversion

**Generator:** Mechanical to electrical energy conversion

**Load:** Current passing through the loop

**Angular Velocity**

**电气 energy being converted to mechanical turning the loop**

**Diagram:**
- Current passing through the loop
- Electrical energy being converted to mechanical turning the loop
- Axis of rotation
- Load
- Magnet
EM Motor/Generator Reciprocity

The loop is turning due to external force
B = Bo in Z direction

Generator: Mechanical to electrical energy conversion
EM Motor/Generator Reciprocity

**Motor:** Electrical to mechanical energy conversion

**Generator:** Mechanical to electrical energy conversion
Applet

- http://www.walter-fendt.de/ph14e/electricmotor.htm
- http://www.walter-fendt.de/ph14e/generator_e.htm
- Good tutorial:
  http://micro.magnet.fsu.edu/electromag/electricity/generators/index.html

Thumb: Conventional direction of current
Forefinger: Magnetic field
Middle finger: Lorentz force
Other Applications

Relay

Generator

Relay

Solenoid
As the loop rotates with an angular velocity $\omega$ about its own axis, segment 1–2 moves with velocity $\mathbf{u}$ given by

$$\mathbf{u} = \hat{n}\omega \frac{w}{2}$$

Also:

$$\hat{n} \times \hat{z} = \hat{x} \sin \alpha.$$

Segment 3-4 moves with velocity $-\mathbf{u}$. Hence:

$$V_{\text{emf}}^m = V_{14} = \int_{1/2}^{3/4} (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} + \int_{3/4}^{1/2} (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

$$= \int_{-l/2}^{l/2} \left[ \left(\hat{n}\omega \frac{w}{2}\right) \times \hat{z} B_0 \right] \cdot \hat{x} \, dx$$

$$= \int_{-l/2}^{l/2} \left[ \left(-\hat{n}\omega \frac{w}{2}\right) \times \hat{z} B_0 \right] \cdot \hat{x} \, dx.$$
Using Magnetic Flux and Faraday's Law

\[ \Phi = \int_B \cdot ds \]
\[ V_{emf} = -\frac{d\Phi}{dt} \]

\[ V_{emf}^m = A\omega B_0 \sin(\omega t + C_0) \quad (V) \]
Faraday’s Law

- The compass in the second coil deflects momentarily and returns immediately to its original position.
- The deflection of the compass is an indication that an electromotive force was induced causing current to flow momentarily in the second coil.
- The closing and opening of the switch cause the magnetic field in the ring to change to expand and collapse respectively.
- Faraday discovered that changes in a magnetic field could induce an electromotive force and current in a nearby circuit.
- The generation of an electromotive force and current by a changing magnetic field is called electromagnetic induction.

http://micro.magnet.fsu.edu/electromag/java/faraday/
Displacement Current

Ampère’s law in differential form is given by

\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = J_c + J_d \]

For arbitrary open surface

\[ \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \int_S \mathbf{J} \cdot d\mathbf{s} + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}. \]

This term is conduction current \( I_c \)

This term must represent a current

Application of Stokes’s theorem gives:

Total Current = \[
\oint_C \mathbf{H} \cdot d\mathbf{l} = I_c + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} \quad \text{(Ampère’s law)}
\]

Conduction current:
(transporting charges)
Displacement current:
(does not transport)

Current Density

Conduction: \( J_c = \sigma \mathbf{E} \)
Displacement: \( J_d = \frac{\partial \mathbf{D}}{\partial t} \)

Note: Convection Current & Conduction Current are difference
**Displacement Current**

\[ \int_C \mathbf{H} \cdot d\mathbf{l} = I_c + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot ds \]

= Total Current = \( I_c + I_d \)

Define the displacement current as:

\[ I_d = \int_S \mathbf{J}_d \cdot ds = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot ds, \]

The displacement current does not involve real charges; it is an equivalent current that depends on \( \frac{\partial \mathbf{D}}{\partial t} \)

\[ J_c = \sigma E \quad \quad J_d = \frac{\partial \mathbf{D}}{\partial t} \]

Note:

If \( \frac{dE}{dt} = 0 \) \( \rightarrow I_d = 0 \)

Maxwell’s contributions: defining the concept of displacement current & unifying time-varying electricity and magnetism
Remember: Conduction Current

Conduction current density:
\[ J = \sigma E \, (A/m^2) \, \text{(Ohm’s law),} \]

Materials: Conductors & Dielectrics

Conductors: Loose electrons \( \rightarrow \) Conduction current can be created due to E field

Dielectrics: electrons are tightly bound to the atom \( \rightarrow \) no current when E is applied

Perfect dielectric: \( J = 0, \)
Perfect conductor: \( E = 0. \)

Conductivity depends on impurity and temperature!

For metals: T inversely proportional to Conductivity!

Perfect Dielectric:
Conductivity = 0 \( \rightarrow \) \( J_c = 0 \)
Perfect Conductor:
Conductivity = INF \( \rightarrow \) \( E = J_c/\sigma = 0 \)
Capacitor Circuit

Given: Wires are perfect conductors and capacitor insulator material is perfect dielectric.

For Surface $S_1$:
$I_1 = I_{1c} + I_{1d}$ ($D = 0$ in perfect conductor)

For Surface $S_2$:
$I_2 = I_{2c} + I_{2d}$
$I_{2c} = 0$ (perfect dielectric)

Conclusion: $I_1 = I_2$

$V(t) = V_0 \cos \omega t$

Note A3b
Given: Wires are perfect conductors and capacitor insulator material is perfect dielectric.

For Surface $S_1$:

\[ I_1 = I_{1c} + I_{1d} \]

\[ I_{1c} = C \frac{dV_C}{dt} = C \frac{d}{dt} (V_0 \cos \omega t) = -CV_0 \omega \sin \omega t \]

\[ I_{1d} = 0 \quad (D = 0 \text{ in perfect conductor}) \]

Remember:

\[ V_{2i} = V_1 - V_2 = -\int_{x_1}^{x_2} E \cdot dl \]

\[ V_{12} = \int_{\frac{1}{2}}^{1} E_m \cdot dl = Ey \cdot d \rightarrow Ey = \frac{V}{d} \]

For Surface $S_2$:

\[ I_2 = I_{2c} + I_{2d} \]

\[ I_{2c} = 0 \quad (\text{perfect dielectric}) \]

\[ E = \hat{y} \frac{V_c}{d} = \hat{y} \frac{V_0}{d} \cos \omega t \]

\[ I_{2d} = \int_{S} \frac{\partial D}{\partial t} \cdot ds \quad \text{Note:} \quad J_d = \frac{\partial D}{\partial t} \]

\[ = \int_{A} \left[ \frac{\partial}{\partial t} \left( \hat{y} \frac{\varepsilon V_0}{d} \cos \omega t \right) \right] \cdot (\hat{y} \, ds) \]

\[ = -\left( \frac{\varepsilon A}{d} \right) V_0 \omega \sin \omega t = -CV_0 \omega \sin \omega t \]

Conclusion: $I_1 = I_2$
Charge Current Continuity Equation

Current I out of a volume is equal to rate of decrease of charge Q contained in that volume:

$$I = -\frac{dQ}{dt}$$

Kirchhoff’s current law:
Algebraic sum of currents following out of a junction is zero
Charge Current Continuity Equation

Current I out of a volume is equal to rate of decrease of charge Q contained in that volume:

\[ I = -\frac{dQ}{dt} = -\frac{d}{dt} \int_V \rho_v \, dV \]

Kirchhoff’s current law:
Algebraic sum of currents following out of a junction is zero

Total Current At junction Is zero
# Maxwell’s Equations – General Set

<table>
<thead>
<tr>
<th>POINT FORM</th>
<th>INTEGRAL FORM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nabla \times \mathbf{H} = \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t}$</td>
<td>$\oint \mathbf{H} \cdot d\mathbf{l} = \int_S \left( \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}$ (Ampère’s law)</td>
</tr>
<tr>
<td>$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$</td>
<td>$\oint \mathbf{E} \cdot d\mathbf{l} = \int_S \left( -\frac{\partial \mathbf{B}}{\partial t} \right) \cdot d\mathbf{S}$ (Faraday’s law; S fixed)</td>
</tr>
<tr>
<td>$\nabla \cdot \mathbf{D} = \rho$</td>
<td>$\int_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho , dv$ (Gauss’s law)</td>
</tr>
<tr>
<td>$\nabla \cdot \mathbf{B} = 0$</td>
<td>$\int_S \mathbf{B} \cdot d\mathbf{S} = 0$ (nonexistence of monopole)</td>
</tr>
</tbody>
</table>
Maxwell’s Equations – Free Space Set

- We assume there are **no charges** in free space and thus, \( J_c = \sigma E = 0 \)

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<tbody>
<tr>
<td>( \nabla \times H = \frac{\partial D}{\partial t} )</td>
<td>( \oint H \cdot dl = \int_s \left( \frac{\partial D}{\partial t} \right) \cdot dS )</td>
</tr>
<tr>
<td>( \nabla \times E = -\frac{\partial B}{\partial t} )</td>
<td>( \oint E \cdot dl = \int_s \left( -\frac{\partial B}{\partial t} \right) \cdot dS )</td>
</tr>
<tr>
<td>( \nabla \cdot D = 0 )</td>
<td>( \oint S D \cdot dS = 0 )</td>
</tr>
<tr>
<td>( \nabla \cdot B = 0 )</td>
<td>( \oint S B \cdot dS = 0 )</td>
</tr>
</tbody>
</table>

**Time-varying E and H cannot exist independently!**
- If \( dE/dt \) non-zero \( \rightarrow dD/dt \) is non-zero \( \rightarrow \) Curl of H is non-zero \( \rightarrow H \) is non-zero

**If H is a function of time \( \rightarrow E \) must exist!**
Maxwell Equations –
Electrostatics and Magnetostatics

Governing equations

- Differential form

\[ \nabla \cdot \mathbf{D} = \rho_v \]
\[ \nabla \times \mathbf{E} = 0 \]

- Integral form

\[ \oint_S \mathbf{D} \cdot d\mathbf{s} = Q \]
\[ \oint_S \mathbf{B} \cdot d\mathbf{s} = 0 \]
\[ \oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \]
\[ \oint_C \mathbf{H} \cdot d\mathbf{l} = I \]
## Boundary Conditions

<table>
<thead>
<tr>
<th>Field Components</th>
<th>General Form</th>
<th>Medium 1 Dielectric</th>
<th>Medium 2 Dielectric</th>
<th>Medium 1 Dielectric</th>
<th>Medium 2 Conductor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tangential E</td>
<td>$\hat{n}_2 \times (E_1 - E_2) = 0$</td>
<td>$E_{1t} = E_{2t}$</td>
<td></td>
<td></td>
<td>$E_{1t} = E_{2t} = 0$</td>
</tr>
<tr>
<td>Normal D</td>
<td>$\hat{n}_2 \cdot (D_1 - D_2) = \rho_s$</td>
<td>$D_{1n} - D_{2n} = \rho_s$</td>
<td></td>
<td>$D_{1n} = \rho_s$</td>
<td>$D_{2n} = 0$</td>
</tr>
<tr>
<td>Tangential H</td>
<td>$\hat{n}_2 \times (H_1 - H_2) = J_s$</td>
<td>$H_{1t} = H_{2t}$</td>
<td></td>
<td>$H_{1t} = J_s$</td>
<td>$H_{2t} = 0$</td>
</tr>
<tr>
<td>Normal B</td>
<td>$\hat{n}_2 \cdot (B_1 - B_2) = 0$</td>
<td>$B_{1n} = B_{2n}$</td>
<td></td>
<td></td>
<td>$B_{1n} = B_{2n} = 0$</td>
</tr>
</tbody>
</table>
Example: Displacement Current

The conduction current flowing through a wire with conductivity \( \sigma = 2 \times 10^7 \) S/m and relative permittivity \( \varepsilon_r = 1 \) is given by \( I_c = 2 \sin \omega t \) (mA). If \( \omega = 10^9 \) rad/s,

\[
I_c = 2 \sin \omega t \text{ (mA)}
\]

Does \( E \) exist? Why? Using Ohm's law; this is not a perfect conductor \( \rightarrow J_c = \sigma E \)

Does \( Id \) exist? If \( E \) exists \( \rightarrow D \) exists (assuming it is time-varying, which is because \( Ic \) is time-varying!) \( \rightarrow Id \) exists

Are \( Ic \) and \( Id \) related? \( E \) is related to \( Jc \); \( Jd \) is defined as change of Electric flux density (\( \varepsilon E \)) in time \( \rightarrow \) They MUST be related!

Find the displacement current.

Note A4
Example

Find $V_{12}$

$U = 5z \, (m/s)$