OVERGROUPS OF IRREDUCIBLE LINEAR GROUPS, I

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1. INTRODUCTION

In work spread over several decades, Dynkin ([4, 3]), Seitz ([10, 11]), and Testerman ([16]) classified the maximal closed connected subgroups of simple algebraic groups. Their analyses for the classical group cases were based primarily on a striking result: If \( G \) is a simple algebraic group and \( \phi : G \to \text{SL}(V) \) is a tensor indecomposable irreducible rational representation, then with specified exceptions the image of \( G \) is maximal among closed connected subgroups of one of the classical groups \( \text{SL}(V) \), \( \text{Sp}(V) \), or \( \text{SO}(V) \). From a slightly different perspective, the question they answered was: Given an irreducible, closed, connected subgroup \( G \) of \( \text{SL}(V) \) for some vector space \( V \), find all possibilities for closed, connected overgroups \( Y \) of \( G \) in \( \text{SL}(V) \).

This question of irreducible overgroups, or the restriction of irreducible modules to subgroups, appears in other contexts as well. In this paper we present some results in the absence of the connectedness requirement for the subgroup. The eventual goal is to classify all possible triples \( (G, Y, V) \) with \( G < \text{Aut}(Y) \) both closed irreducible subgroups of \( \text{SL}(V) \), \( Y \neq \text{SL}(V), \text{SO}(V) \), or \( \text{Sp}(V) \), and \( Y \) a simple group of classical type. In this paper and [5], we give complete results for the case when \( G \) is not connected but has simple identity component \( X \), and \( Y \)-high weight and \( X \)-high weights of \( V \) are restricted. Specifically, the papers are concerned with the proof of Theorem 1 below.

Let \( G \) be a non-connected algebraic group with simple identity component \( X \). Let \( V \) be an irreducible \( KG \)-module with restricted \( X \)-high weight(s).

**Theorem 1.** Let \( Y \) be a simple algebraic group of classical type such that \( X < Y < \text{SL}(V) \), \( G \leq \text{Aut}(Y) \), and \( V|_Y \) is irreducible with restricted high weight. Then \( Y = \text{SO}(V) \), \( Y = \text{Sp}(V) \), or \( (X, Y, V) \) appears in Table 1 or Table 2.

If \( G \) has simple identity component \( X \), then \( G \leq \text{Aut}(X) \). Since we require that \( G \neq X \), we therefore may restrict our attention to \( X \) of type \( A_m, D_m \), or \( E_6 \).

The group \( Y \) is of classical type and so has a natural module \( W \). Some simply connected cover \( \tilde{Y} \) of \( Y \) acts irreducibly on \( W \). Let \( \tilde{X} \) be the preimage of \( X < \tilde{Y} \) under the projection \( \tilde{Y} \to Y \). Then \( \tilde{X} = X \cdot Z \) for some cover \( X \) of \( Y \) and some \( Z \subseteq \text{Z}(\tilde{Y}) \). Now we replace \( Y \) by \( \tilde{Y} \) and \( X \) by \( \tilde{X} \). If \( u \) is an outer automorphism of \( X \), then the action of \( u \) can be extended to \( \tilde{X} \); if \( u \in Y \) let \( \tilde{u} \) be a preimage of \( u \) under the projection \( \tilde{Y} \to Y \). If \( u \) is an outer automorphism of \( Y \), it is again possible to extend \( u \) to \( \tilde{u} \in \text{Aut}(\tilde{Y}) \). Now replace \( u \) by \( \tilde{u} \). As we will make use only of the action of \( u \) on 1-spaces in \( Y \)- and \( X \)-modules, the possible extension by elements of \( Z(\tilde{Y}) \) does not concern us. So we assume henceforth that \( Y \) is simply connected, and that \( X \) and \( Y \) act on \( W \).

The analysis is different depending on whether \( X \) acts reducibly or irreducibly on \( W \). We settle the reducible case in this paper, and the irreducible in [5]. Also, we will assume in [5] that the involutory graph automorphism of \( X \), if it is in \( G \), also acts on \( W \) (though it need not be in \( Y \)). We deal with the case when it does not act on \( W \) in the final section of this paper.

If \( V|_X \) is irreducible, then we are in the case examined by Seitz in [10], with the additional condition that \( X \) have an outer automorphism which acts on \( V \). We examine Table 1 of that paper, and find that we have such a situation in the examples there labelled \( I_4, I_5, I_6 \) for \( n = 3 \), \( II_1, S_1, S_8 \) (here we could take \( G = X(t), G = X(s), \) or \( G = X(t,s) \), where \( t, s \) are outer automorphisms of \( X \) of order 2 and 3 respectively), and \( \text{MR}_4 \). Henceforth we shall assume that \( V|_X \) is not irreducible.

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1.1. Notation and Conventions. All structures are assumed to be constructed over the same algebraically closed field $K$, of characteristic $\rho \geq 0$. Throughout, $X$ will denote a simple algebraic group over $K$ admitting an outer automorphism (so $X$ is of type $A_m, D_m, E_6$). A fixed standard graph automorphism of order 2 will be denoted by $t$, and if $X$ has an outer automorphism of order 3 (i.e. if $X = D_4$), we will fix one and denote it by $s$. Thus $G = X(t)$ except possibly when $X = D_4$, in which case we also consider $G = X(s)$ and $G = X(s,t)$.

We let $B_X$ be a fixed $t$-stable Borel subgroup of $X$, containing a fixed $t$-stable maximal torus $T_X$. Define sets of simple roots $\{\beta_1, \beta_2, \ldots, \beta_n\} = \Pi(X)$ and fundamental dominant weights $\{\delta_1, \ldots, \delta_m\}$ with respect to $T_X$ and $B_X$, but with the opposite of the standard convention: $B_X = U_X T_X$ where $U_X = \prod U_{-\alpha}$ for $\alpha \in \Sigma^+(X)$. Then for $J \subseteq \Pi(X)$, $P_X$ is the opposite of the standard parabolic corresponding to $J$. We assume the $\delta_i$ are numbered so that $\delta_i$ corresponds to $\beta_i$ for every $i$. The set of roots of $X$ is $\Sigma(X)$; the set of positive roots $\Sigma^+(X)$.

The group $Y$ will be a simple algebraic group over $K$ of classical type and rank $n (A_n, B_n, C_n$ or $D_n)$, such that $X < Y$ and $G \leq \text{Aut}(Y)$. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} = \Pi(Y)$ be a set of simple roots of $Y$, and $\{\lambda_i\}$ the set of fundamental dominant weights such that $\lambda_i$ corresponds to $\alpha_i$. The set of roots of $Y$ is $\Sigma(Y)$; the set of positive roots $\Sigma^+(Y)$. Notation and conventions similar to those used for $X$ are used for parabolic subgroups of $Y$.

For a group $H$ acting on a module $M$, $[M, H]$ will denote the $t$-fold commutator of $H$ with $M$.

The $K$-vector space $V$ is assumed to be a restricted irreducible $Y$-module with high weight $\lambda = \sum a_i \lambda_i$, such that $V$ is irreducible as a $G$-module but not as an $X$-module (see the comment at the end of the previous subsection). We assume that the $T_X$-high weights of $V$ are restricted as well. So if $G = X(t)$, then $V|_X = V_1 \oplus V_2$, where each of $V_1, V_2$ is a restricted irreducible $X$-module.

The natural module for $Y$ will be denoted by $W$. If $W$ is irreducible as an $X$-module, then $\delta$ will denote its $T_X$-high weight. We will always assume that $Y$ is the smallest of $\text{SL}(W), \text{SO}(W), \text{Sp}(W)$ containing $X$. To justify this assumption, we must eliminate the situation when $X < Y < \text{SL}(W), Y = \text{SO}(W)$ or $\text{Sp}(W)$, and $V$ is a reducible $Y$-module, but an irreducible $G$-module. Assume we have such a situation.

If $X = D_4$ and $s \in G$, then $X(s) < Y$ because $X$ has no outer automorphisms of order 3. In this case let $\tilde{X} = X(s)$; otherwise set $\tilde{X} = X$. Now $V_1|_X = V_1 \oplus V_2$ and $V_2|_Y = V_1 \oplus V_2$ for some irreducible $\tilde{X}$- and $Y$-modules $V_1$ and $V_2$. Now if $\tilde{X} = X$, then since $X \neq Y$, the triple $(V_1, X, Y)$ must appear in the list in [10]. Here, though, we have the extra hypotheses that $Y \neq A_4$: $X$ has a graph automorphism not fixing $V_1$ (or $V_2$); both $V_1$ and $V_2$ appear in the table; and $Y$ has an outer automorphism ($G \leq \text{Aut}(Y)$) $t \not\in Y$ since $V_1|_Y$ is not irreducible). There are no entries in [10] which satisfy all these conditions.

Similarly, if $\tilde{X} = D_4(s)$, then $(V_1, D_4(s), Y)$ would appear in Table 2, with $G = D_4(s)$; there are no such examples there. So our standing assumption that $Y$ is the smallest of $\text{SL}(W), \text{SO}(W), \text{Sp}(W)$ containing $X$ is justified.

Except for the last section, we will assume that $G$ acts on $W$ (X clearly acts on $W$ since $X \leq Y$, but $t$ might not act on $W$). When $t \in G$ acts on $W$, $X(t)$ fixes a nondegenerate bilinear form on $W$ ([14, Lemma 79]); but in characteristic 2, it is conceivable that $X$ fixes a quadratic form which $t$ does not. Then we would have $X \leq \text{SO}(W); X(t) \leq \text{Sp}(W)$. Let $q$ be the quadratic form defining $\text{SO}(W)$; let $q'$ be the quadratic form given by $q'(w) = q(w')$ for $w \in W$. Both forms are $X$-invariant.

Lemma 1.1. With the above setup, $q = q'$, so $q$ is $t$-stable and $X(t) \leq \text{O}(W)$.

Proof. Let $R = \{w \in W | q(w) = q'(w)\}$. It is clear that $R$ is a subspace of $W$ and that $R$ is $X$-invariant. But $X$ acts irreducibly on $W$, so $R = 0$ or $R = W$. For a $T_X$-maximal vector $w^+ \in W$, $q(w^+) = 0 = q'(w^+)$ (let $h \in T_X$ such that $hw^+ = sw^+$ for some $s \in K^*$ such that $1 \neq s^2$ — then $q(w^+) = q(hw^+) = s^2 q(w^+)$, so $q(w^+) = 0$; similarly for $q'$). So $R = W$, which implies $q = q'$.

We label Dynkin diagrams for the groups we will be dealing with as follows, and we always number fundamental roots and fundamental dominant weights to agree with this labelling:

\begin{align*}
A_1: & 1 \quad 2 \quad l-1 \quad l \\
B_1: & 1 \quad 2 \quad l-1 \quad l
\end{align*}
2. $Q \gamma$-LEVELS AND EMBEDDINGS OF PARABOLICS

In this section we introduce important facts about the “commutator series” of a module of a simple algebraic group.

**Lemma 2.1.** If $H$ is a simple algebraic group whose root system has only one root length, then restricted irreducible $H$-modules are tensor indecomposable (in particular, restricted irreducible $X$-modules are tensor indecomposable).

*Proof.* This is part of 1.6 of [10].

**Lemma 2.2.** Let $M$ be an irreducible restricted $H$-module with high weight $\gamma$ for some simple algebraic group $H$. Let $P$ be a proper parabolic subgroup of $H$, with $P = QL$ a Levi decomposition. Then $M/[M, Q]$ is irreducible for $L$ and for $L' = [L, L]$, with $T_{L'}$-high weight $\gamma|_{T_{L'}}$.

*Proof.* This is 1.7 and 2.1 of [10].

Let $H, M, \gamma,$ and $P$ be as in the last lemma. Let $\{\varepsilon_i\}$ be the set of fundamental roots of $H$.

**Definition 2.3.** Let $\mu$ be a weight of $M$, say $\mu = \gamma - \sum c_i \varepsilon_i$, with each $c_i \geq 0$. The $Q$-level of $\mu$ is $\sum c_j$, where the sum ranges over those $j$ for which $\varepsilon_j \in \Pi(H) - \Pi(L')$. The $Q$-level $l$ of $M$ is the sum of weight spaces for weights having $Q$-level $l$ and is denoted $M_l$.

**Lemma 2.4.** $H, M, \gamma, P$ as above. If $H$ is simply laced or if $p > 2$ ($> 3$ for $H = G_2$), then

1. $[M, Q] = \oplus M_{\mu}$, the sum taken over those weights $\mu$ having $Q$-level at least $l$.
2. $[M, Q^l]/[M, Q^{l+1}] \cong M_l$.
3. $\dim([M, Q^l]/[M, Q^{l+1}]) \leq s \cdot \dim([M, Q^{l-1}]/[M, Q^l])$, where $s$ is the number of positive roots $\beta$ such that $U_{-\beta} \leq Q$ and $\beta = \varepsilon_i + \beta'$ for some $\varepsilon_i \in \Pi(H) - \Pi(L')$, with $\beta' = 0$ or a sum of roots in $\Pi(L')$.
4. $\dim([M, Q^l]/[M, Q^{l+1}]) \leq \dim(Q) \cdot \dim([M, Q^{l-1}]/[M, Q^l])$.

*Proof.* This is 2.3 of [10].

We will write $M'^{(Q \gamma)}$ for the quotient $[M, Q^{l-1}]/[M, Q^l]$.

**Lemma 2.5.** Let $H = A_1$; let $c$ be an integer such that $0 < c < p$; and let $\gamma_1, \gamma_1$ be the “end” fundamental dominant weights for $H$. The irreducible module $M$ having high weight $c\gamma_1$ or $c\gamma_1$ has all weight spaces of dimension 1; in particular, $\dim(M) = (l + c)!/l!c!$.

*Proof.* This is 1.14 of [10].

We will occasionally use the Weyl character formula for dimensions of Weyl modules.

Finally, we claim that when $X$ acts irreducibly on $W$, we may assume $W$ is in fact restricted as an $X$-module.

**Lemma 2.6.** If $X$ acts irreducibly on $W$, then as an $X$-module, $W$ has a restricted high weight.
Proof. By Steinberg’s tensor product theorem ([13]), $W = W_1^q_1 \otimes W_2^q_2 \otimes \cdots \otimes W_r^q_r$ where the $W_i$ are restricted irreducible $X$-modules and $q_1, \ldots, q_r$ are distinct powers of $p$. Let $X_i$ be the isometry group of $W_i^q_i$. Then the embedding of $X$ in $Y$, which is given by the action on $W$, factors through an embedding $X \mapsto X_1 X_2 \cdots X_r \leq Y$ given by $x \mapsto (x^q_1, x^q_2, \ldots, x^q_r)$. But then the action of $X$ on $V$ factors through this same embedding (since the action of $X$ on $V$ is given by the embedding of $X$ in $Y$). Notice that each $X_i$ must act nontrivially on $Y$ since $Y$ is simple. Now if $V|_{X_1 X_2 \cdots X_r}$ is irreducible, then as an $X$-module, $V$ has a high weight given by the restriction of the $X_1 X_2 \cdots X_r$-high weight to $T_X$; but this gives a $T_X$-high weight of the form $q_1 \gamma_1 + \cdots + q_r \gamma_r$ for some non-zero restricted $T_X$-weights $\gamma_i$. This is impossible unless $r = 1$ and $q_1 = 1$, as the $T_X$-high weights of $V$ are restricted. Similarly, if $V$ is reducible as an $X_1 X_2 \cdots X_r$-module and there is an $i$ such that $q_i \neq 1$, then let $V_i$ be a $X_1 X_2 \cdots X_r$-subquotient of $V$ on which $X_i$ acts nontrivially. Then as an $X$-module, $V_i$ has a non-restricted high weight as above, again a contradiction. \[\square\]

Now let $P_X$ be a parabolic subgroup of $X$, and $P_X = Q_X L_X$ a Levi decomposition with $T_X \leq L_X$ (if $P_X$ is $t$-stable, choose $L_X$ to also be $t$-stable). Assume that $X$ acts irreducibly on $W$ with high weight $\delta$, which is restricted by the Lemma above. To give a construction of a parabolic subgroup $P_Y$ of $Y$ (with $P_Y = Q_Y L_Y$ a Levi decomposition) such that $P_X \leq P_Y$, $Q_X \neq Q_Y$, $L_X \leq L_Y$. Let $Z = Z(L_X)^\circ$.

Lemma 2.7. The stabilizer in $Y$ of the commutator series

$$W > [W, Q_X] > [W, [Q_X, Q_X]] > \cdots > 0$$

is a parabolic subgroup $P_Y$ of $Y$ satisfying the following:

1. $P_X \leq P_Y$ and $Q_X \neq Q_Y = R_0(P_Y)$.
2. $L_Y = C_Y(Z)$ is a Levi factor of $P_Y$ containing $L_X$.
3. If $T_Y$ is a maximal torus of $Y$ containing $T_X$, then $T_Y \leq L_Y$.

Proof. Since $X$ acts irreducibly on $W$, we can apply the $Q_X$-level construction above to $W$ and $P_X$. Suppose $Y = Sp(W)$ or $SO(W)$. Then $W \cong W^*$ and we identify these modules via the inner product. So the fixed point sets $W(Q_X)$ and $W^*(Q_X)$ are equal, and a trivial calculation shows that $W^*(Q_X)$ is the annihilator in $W^*$ of $[W, Q_X]$. But from 1.2 in [10] it follows that $W(Q_X) = [W, Q_X^k]$, where $k$ is minimal with respect to $[W, Q_X^{k+1}] = 0$. By induction, $[W, Q_X^k]$ annihilates $[W, Q_X^{k+1}]$.

By Lemma 2.4, $[W, Q_X^k]/[W, Q_X^{k+1}] \cong \sum W_{\mu}$, with the sum taken over those weights $\mu$ of $Q_X$-level $l$, so by the last paragraph we have $(W, W_j) = 0$ unless $i + j \leq k$. Let $c = [(k - 1)/2]$. Then $0 < W_k \leq \cdots \leq W_1 \oplus \cdots \oplus W_{k-1}$ is a flag of totally isotropic subspaces of $W$, so its stabilizer $P_Y$ in $Y$ is a parabolic subgroup. But our discussion of weights shows this flag is just the flag $0 < [W, Q_X^k] < [W, Q_X^{k-1}] < \cdots < [W, Q_X^1]$, and if we adjoin annihilators, we obtain the full commutator series $0 < [W, Q_X^k] < \cdots < [W, Q_X] < W$. Hence $P_Y$ is the stabilizer of this commutator series. Let $Q_Y = R_0(P_Y)$.

Let $Z = Z(L_X)^\circ$. Let $L$ be a Levi factor of $P_Y$ containing $Z$, and let $L_Y = C_Y(Z) > L_X$. Then $Z$ induces scalars on $W/[W, Q_X]$ since this module is an irreducible $L_X$-module, and so $[Z, L] \leq L$ has a trivial action on $W/[W, Q_X]$, which implies $[Z, L] \leq Q_Y$. But $Q_Y \cap L = 1$, so $L \leq C_Y(Z) = L_Y$. This implies $L = L_Y$ as $\{L\}$ is a maximal reductive subgroup of $P_Y$. Let $T_Y$ be a maximal torus of $Y$ containing $T_X$. Then $T_Y \leq C_Y(T_X) \leq C_Y(Z) = L_Y$.

Note that $[W, Q_X^k] = [W, Q_X^{k-1}]$ for every $k$ by construction. If $u \in U_{-a}$ for $a \in \Pi(X) - \Pi(L_X)$, then $uW_{-e_1} - e_1 - e_2 - \cdots \leq \sum_{t \geq 0} W_{-e_1} - e_1 - e_2 - \cdots - e_t$. So $Q_X \leq Q_Y$. Since $P_X$ stabilizes each factor in the flag, $P_X \leq P_Y$.

If $Y = SL(W)$, the argument is easier: The flag $0 < [W, Q_X^k] < [W, Q_X^{k-1}] < \cdots < [W, Q_X] < W$ determines a parabolic subgroup $P_Y$ and the above arguments hold. \[\square\]

We give more information about this embedding for particular groups $X$ and parabolic subgroups $P_X$ below and in subsequent sections. For the next two Lemmas, we assume that $t \in G$ (where $t$ is the fixed outer automorphism of $X$) and $V = V_1 \oplus V_2$, with $V_1, V_2$ irreducible $X$-modules.

Lemma 2.8. If $P_X$ is a $t$-stable parabolic subgroup of $X$ and $P_X$ is embedded in a parabolic subgroup $P_Y$ of $Y$ as above, then $P_Y$ is likewise $t$-stable.
Proof. This is clear if \( t \) acts on \( W \), for then each subspace \( W_i \) of \( W \) is \( t \)-stable (since, if \( X \) is of type \( A_m \) for example, \( W_{k \in c_1 \mathbb{B}_1 - c_2 \mathbb{B}_2 - \cdots - c_m \mathbb{B}_m} = W_{k \in c_1 \mathbb{B}_1 - c_2 \mathbb{B}_2 - \cdots - c_m \mathbb{B}_m} \)). So \( P_Y(t) \) is the stabilizer in \( W \) of the same flag as is \( P_Y \); i.e. \( P_Y = P_Y(t) \).

Now assume \( t \) does not act on \( W \). Then, since \( t \) acts on \( Y \), we have \( Y = A_n \) (as outer automorphisms of \( D_n \) preserve the natural module). Let the \( Q_X \)-level of the \( T_X \)-low weight of \( W \) be \( k \) as above. The dimensions of \( Q_X \)-levels of \( W \) are symmetric about \( k/2 \); that is, \( \dim(W_i) = \dim(W_{k-i}) \), since they are interchanged by a representative in \( X \) of the long word of the Weyl group. This means that \( P_Y(t) \) is symmetric; i.e. \( \Pi(L_{\beta}) \) is preserved under the automorphism of the Dynkin diagram of \( Y \). So there is a graph automorphism \( \sigma \) of \( Y \) which preserves \( P_Y \); since all graph automorphisms may be written as the product of \( t \) with an element of \( Y \), there is a \( g \in Y \) such that \( t = \sigma g \). Then \( P_Y(t) = P_Y(t)^{t \sigma} = P_Y \). So \( P_Y(t) \) and \( P_Y \) are conjugate.

Since \( P_Y(t) \) is the stabilizer of the flag \( W > [W,Q_Y] > \ldots P_Y = P_Y(t) \) is the stabilizer of \( W > [W,Q_Y] \}\( Q_Y \) is the stabilizer of \( W > [W,Q_Y] \}\( 2 > \ldots \). Also, \( P_Y \neq P_Y(t) \) so \( Q_X \leq Q_X \) \( Q_X \). This gives \( [W,Q_Y^i] = [W,Q_Y^j] \leq [W,Q_Y^j] \) for every \( i \). But \( \dim([W,Q_Y^i]) = \dim([W,Q_Y^i]^j) \); so in fact \([W,Q_Y^i]^j = [W,Q_Y^i]^j \) for every \( i \). But this implies that \( P_Y(t) \) and \( P_Y \) are the stabilizers of the same flag in \( W \), or \( P_Y = P_Y(t) \).

Lemma 2.9. If \( P_X \) is a \( t \)-stable parabolic subgroup of \( X \) and \( P_X \) is embedded in a parabolic subgroup \( P_Y \) of \( Y \) as above, then \( V/V[Q_Y] = V/V[Q_X] = V_1/[V_1,Q_X] \oplus V_2/[V_2,Q_X] \).

Proof. The involution \( t \) interchanges the two \( T_X \)-high weight 1-spaces \( \langle v_1 \rangle \subseteq V_1 \) and \( \langle v_2 \rangle \subseteq V_2 \) of \( W \). We have \([V_i,Q_X] \leq [V_i,Q_Y] \) since \( Q_X \leq Q_Y \), and \( V'[V_i,Q_X] = V_1/[V_1,Q_X] \oplus V_2/[V_2,Q_X] \), with each summand an irreducible \( L_X \)-module by Lemma 2.2.

Since \( L_X \) is contained in \( L_Y \) and \([V_j,Q_Y] \) is an irreducible \( L_Y \)-module, either \( (V/[V_i,Q_Y]) \leq V_i/[V_i,Q_X] \) and \([V_j,Q_Y] \leq V_1/[V_1,Q_X] \oplus V_2/[V_2,Q_X] \) for \( i = 1 \) or 2, or \( (V/[V_i,Q_Y]) \leq V_1/[V_1,Q_X] \oplus V_2/[V_2,Q_X] \). If the former, say \( (V/[V_i,Q_Y]) = V_1/[V_1,Q_X] \oplus V_2/[V_2,Q_X] \), then \( v_2 \in [V_i,Q_Y] \) (notice that \( V_1/[V_1,Q_X] \neq V_2/[V_2,Q_X] \) as \( L_X \)-modules because \( T_X \leq L_X \) and \( t \) acts differently on a high weight vector \( v_1 \) than on a high weight vector \( v_2 \) in \( V_2 \), and \( v_1 \) has a non-zero image in \( V_i/[V_i,Q_Y] \)). But \( Q_Y \) is \( t \)-stable by the last lemma \( Q_Y \) is a characteristic subgroup of \( P_Y \), so \([V_i,Q_Y] \) is \( t \)-stable. This would imply \( \langle v_2 \rangle = \langle v_1 \rangle \subseteq [V_i,Q_Y] \), which is impossible. So \( V/[V_i,Q_Y] = V_1/[V_1,Q_X] \oplus V_2/[V_2,Q_X] \).

Let \( P_Y = L_YQ_Y \) be a parabolic subgroup of \( Y \). For each \( \gamma \in \Pi(Y) - \Pi(L_{\beta}) \), we define a certain normal subgroup \( K_{\gamma}^\beta \) of \( P_Y \), as in [10, page 44]: Let \( \Sigma(Y) \) denote the set of roots in \( \Sigma(Y) \) having \( \gamma \)-coefficient \(-1 \) and \( 0 \)-coefficient for other roots in \( \Pi(Y) - \Pi(L_{\beta}) \). Then \( K_{\gamma}^\beta \) is the product of those \( T_Y \)-root subgroups \( U_{\beta} \) for \( \beta \in \Sigma(Y) - \Sigma(L_{\beta}) - \Sigma(Y) \). From the commutator relations it follows that \( K_{\gamma}^\beta \) is normal in \( P_Y \) and we let \( Q_{\gamma}^\beta = K_{\gamma}^\beta/K_{\gamma} \). This construction also applies to a parabolic subgroup \( P_X \) of \( X \). In particular, if \( P_X \) is a maximal parabolic subgroup corresponding to \( \alpha \in \Pi(X) \), then set \( Q_{\alpha}^\alpha = Q_X/K_{\alpha}^\alpha \), where \( K_{\alpha}^\alpha \) is the product of those \( T_{\alpha} \)-root subgroups corresponding to \( \alpha \) having \( \alpha \)-coefficient strictly less than \(-1 \).

Lemma 2.10. If \( P_X = Q_XL_X \) is a maximal parabolic subgroup corresponding to \( \alpha \in \Pi(X) \), then:

1. \( K_{\alpha}^\alpha = [Q_X,Q_X] \).
2. \( Q_{\alpha}^\beta \) is an irreducible \( L_X \)-module with \(-\alpha \) as its \( L_{\alpha} \)-high weight.

Proof. See 3.2 in [10] (remembering that \( X \) is of type \( A_m, D_m \), or \( E_6 \)).
Lemma 2.11. If $V$, $P_X = L_X Q_X$, $P_Y = L_Y Q_Y$, and $L_i$ are as above with $P_X$ $t$-stable, then only one $L_i$ acts nontrivially on $V/[V,Q]$. 

Proof. By Lemma 2.9, $V/[V, Q_Y] = V/[V, Q_X] = V_1/[V_1, Q_X] \oplus V_2/[V_2, Q_X]$. Let $V^i$ be the obvious module for $L_i$ as above (i.e. if $L_i$ corresponds to $\{\alpha_i, \alpha_{i+1}, \ldots, \alpha_k\} \subseteq \Pi(Y)$, then $V^i$ is the $L_i$-module with high weight $a_i \lambda_i + a_{i+1} \lambda_{i+1} + \cdots + a_k \lambda_k$). Then $\otimes V^i_{L_i} = V/[V, Q_Y] = V_1/[V_1, Q_X] \oplus V_2/[V_2, Q_X] = V^i_1 \oplus V^i_2$, with the latter two (restricted) irreducible $L_i^X$-modules. Notice that $V/[V, Q_Y]$ has no tensor decomposable $L_i^X$-submodules, since the only $L_i^X$-submodules of $V/[V, Q_Y]$ are isomorphic to $V^i_1$ or $V^i_2$, which are both irreducible $L_i^X$-modules, and Lemma 2.1 implies that no irreducible $L_i^X$-module is tensor decomposable.

None of the $V^i$ can be reducible as $L_i^X$-modules: Assume $V^1$ has an $L_i^X$-submodule $V'$. Then $V^1 \otimes V^2 \otimes \ldots$ is a proper tensor decomposable $L_i^X$-invariant subspace of $V/[V, Q_Y]$, which is impossible by the above. So each of the $V^i$ is an irreducible $L_i^X$-module; they are restricted as well, since the sum of their $L_i^X$-high weights must be an $L_i^X$-high weight in $V/[V,Q]$, and these are both restricted by assumption.

Similarly, assume there are more than two $V^i$. Then $V^1 \otimes V^2$ is not an irreducible $L_i^X$-module by Steinberg's tensor product theorem ([13]) (since there is no twist in the embedding $L_i^X \hookrightarrow L_i$), so it must have a proper $L_i^X$-invariant subspace $V'$. But then $V^1 \otimes V^2 \otimes \ldots$ is a proper tensor decomposable $L_i^X$-invariant subspace of $V/[V,Q]$, which is again a contradiction.

So assume two $V^i$ are nontrivial; we have $V^1 \otimes V^2 = V_1 \oplus V_2$, all restricted irreducible $L_i^X$-modules. Let $\lambda^1$ be the $T_x \cap L_i^X$-high weight of $V^1$; $\gamma^1$ the $T_x \cap L_i^X$-high weight of $V^1$. The Levi factor $L_f = C_Y(Z(L_x)^2)$ is $t$-stable since $L_x$ is; hence $L_f^X$ is $t$-stable. Note that each $L_i$ has rank greater than 1 (since $L_i^X$ projects nontrivially to $L_i$). Thus the subsystem groups $L_1$ and $L_2$ are either interchanged by $t$ (if $Y$ has type $A_m$ and $t \notin Y$) or fixed by $t$ (if $t \in Y$ or $Y$ has type other than $A_n$). Let $v^1$ (respectively $v^2$) be a $T_{i_1}^X$-high weight vector of $V^1$ (respectively $V^2$), with respect to $B_X \cap L_i^X = B_{i_1}^X$ (a Borel subgroup of $L_i^X$). Then $v^1 \otimes v^2$ is a $T_{i_2}^X$-high weight vector of $V^1 \otimes V^2 = V/[V, Q_Y]$, and since $Z = Z(L_x)^2$ induces scalars on $V/[V, Q_Y]$, $v^1 \otimes v^2$ is in fact a $T_{i_2}^X$-high weight vector in $V/[V, Q_Y]$.

Assume that $Y = A_n$ and $t \notin Y$, so $L^1_i = L_2$. Then $\lambda$ is symmetric with respect to the graph automorphism (since $t$ acts on $V$), so $(v^1)^t \in K v^2$ and $(v^2)^t \in K v^1$ (as $t$ stabilizes $B_{i_2}^X$ and $T_{i_2}^X$).

If $L_1$ and $L_2$ are fixed by $t$, then $t$ preserves each of $V_1$ and $V_2$; hence, as $t$ stabilizes $B_{i_2}^X$ and $T_{i_2}^X$, it stabilizes $K v^1$ and $K v^2$. So in any case, $(v^1 \otimes v^2)^t \in K (v^1 \otimes v^2)$, which is a contradiction as the two $T_x$-high weight 1-spaces in $V^1 \otimes V^2$ are interchanged by $t$. So one of the $V^i$ must be trivial. 

One final crucial lemma, due to Suprunenko:

Lemma 2.12. Let $H$ be a simple algebraic group of type $A_1$, and $M$ an irreducible $H$-module with restricted high weight $\gamma$. Let $W$ be the Weyl module for $H$ with high weight $\gamma$. Assume that $\mu$ is a weight such that $W_\mu \neq 0$. Then $M_\mu \neq 0$.

Proof. This is the result of [15].

3. THE $D_n < B_n$ CASE

In Section 4, we will present the proof of Theorem 1 for the case when $W|_X$ is reducible. It turns out that the hardest case will be when $X = D_n < B_n = Y$, so we do that case first. So we assume $X = D_n$, $G = X(t)$, $Y = B_n$, with $X$ contained in $Y$ in the usual way.

Notation is as before. For $H$ an algebraic group, let $L(H)$ denote the Lie algebra of $H$. For a simple Lie algebra with root system $\Phi$ having basis $\{\alpha_1, \ldots, \alpha_m\}$, we use the standard Chevalley basis $\{e_{\alpha_i}, f_{\alpha_i}, h_i|\alpha \in \Phi^+, 1 \leq i \leq m\}$, satisfying the usual relations (in particular, $[e_{\alpha_i}, f_{\alpha_i}] = h_i$). The usual ordering on weights will be denoted by $\succ$.

Throughout this section, $X$ is a simple algebraic group over $K$ of type $D_n$, embedded in $Y = B_n$ in the usual way (as the derived group of the stabilizer of a 1-space).

First we prove a proposition for irreducible representations of simple Lie algebras. The methods used in the proof of the proposition have an interesting application to the representation theory of the symmetric groups; see [6].
3.1. Lie Algebra Representations. Let $V$ be an irreducible $L(B_n)$-module, with high weight $\lambda = a_1\lambda_1 + a_2\lambda_2 + \cdots + a_n\lambda_n$, $a_n \neq 0$ ($\lambda_i$ the fundamental dominant weight corresponding to the root $\alpha_i$), and high weight vector $\nu^+$. Assume $p \neq 2$. For the weight $\mu = \lambda - (\alpha_i + \cdots + \alpha_k)$ ($k \geq i$), $V_\mu$ is spanned by vectors of the form

$$f_{\alpha_1 + \cdots + \alpha_k}f_{\alpha_{i+1}} + \cdots + f_{\alpha_{i+1} + \cdots + \alpha_k}\nu^+, \quad (\dagger)$$

where every $f_\beta$ involves some $\alpha_j$ such that $a_j \neq 0$. For such a $k \geq i$, let $V_{i,k}$ denote the span of all of the above terms except for $f_{\alpha_1 + \cdots + \alpha_k}\nu^+$; so $V_{i,k} \subseteq V_\mu$.

**Proposition 3.1.** Let $V$ and $\lambda$ be as above. Let $a_i$, $a_m$ be non-zero labels with $m > i$. If $f_{\alpha_1 + \cdots + \alpha_k}\nu^+ \in V_{i,m}$ for all $i \leq r < m$, then $f_{\alpha_1 + \cdots + \alpha_k}\nu^+ \in V_{i,j}$, where $a_j$ is the first non-zero label after $a_i$.

**Proof.** Assume the hypotheses. If $j = m$, the Proposition is vacuous; so assume there are non-zero coefficients between $a_i$ and $a_m$; we proceed by induction on the number of such non-zero coefficients. Let $a_k$ be the last non-zero coefficient before $a_m$.

For the calculations below, recall that since $\nu^+$ is a maximal vector, $e_{\beta}\nu^+ = 0$ for any $\beta \in \Sigma^+$, and that $e_\beta$ and $f_\delta$ commute whenever $\beta - \delta \not\in \Sigma$. If $\beta - \delta \in \Sigma$, then $e_\beta f_\delta = f_\delta e_\beta + d e_{\beta-\delta}$ for some $d = N(\beta, -\delta) \in \mathbb{Z}$ (where for $\alpha \in \Sigma^-$, $e_\alpha = f_\alpha$).

**Lemma 3.2.** $f_{\alpha_1 + \cdots + \alpha_k}\nu^+ \in V_{r,k}$ for all $i \leq r < k$.

**Proof.** Assume that there is an $r \in [i, k-1]$ such that $f_{\alpha_1 + \cdots + \alpha_k}\nu^+ \notin V_{r,k}$.

To say $f_{\alpha_1 + \cdots + \alpha_m}\nu^+ \in V_{r,m}$ is to say that there is a non-zero linear combination

$$0 = f_{\alpha_1 + \cdots + \alpha_m}\nu^+ + \sum_{(p_1, \ldots, p_l)} d_{(p_1, \ldots, p_l)} f_{p_1} \cdots f_{p_l} \nu^+, \quad (*)$$

where $f_{p_1} \cdots f_{p_l} \nu^+$ is a term of type $(\dagger)$, with $s \geq 2$ and $\sum p_j = \alpha_1 + \cdots + \alpha_m$.

Now consider those summands of the right-hand side of $(*)$ which give a multiple of $f_{\alpha_1 + \cdots + \alpha_k}\nu^+$ when the product $e_\gamma_1 \cdots e_\gamma_l$ is applied, where $\gamma_1 = \alpha_{k+1} + \cdots + \alpha_{k+l}, \gamma_2 = \alpha_{k+l+1} + \cdots$ and $\sum \gamma_l = \alpha_{k+1} + \cdots + \alpha_m$. Since each $f_{p_j}$ must involve some $\alpha_k$ with $a_j \neq 0$, these terms are exactly the

$$F_i \nu^+ = f_{\alpha_1 + \cdots + \alpha_k}f_{\alpha_{i+1} + \cdots + \alpha_m}\nu^+, \quad 0 \leq l \leq m - k,$$

since all the coefficients between $a_k$ and $a_m$ are 0. Notice that such a product of $e_\gamma$’s applied to a summand of $(*)$ gives a multiple of a generator of type $(\dagger)$.

Let the coefficient of $F_i \nu^+$ in $(*)$ be $b_l = d_{(\alpha_1 + \cdots + \alpha_k, \alpha_{i+1} + \cdots + \alpha_m)}$; note that $b_{m-k} = 1$. Since by our assumption $f_{\alpha_1 + \cdots + \alpha_k}\nu^+$ is not a combination of other terms $(\dagger)$ which appear in the $\lambda - (\alpha_r + \cdots + \alpha_k)$ weight space, any $e_\gamma_1 \cdots e_\gamma_l$ ($\gamma_l$ as in the last paragraph) must kill $\sum b_l F_i \nu^+$. In particular, $E_s = e_{\alpha_{k+1} + \cdots + \alpha_{k+l}}, e_{\alpha_{k+l+1} + \cdots + \alpha_m}, \quad 0 \leq s \leq m - k,$

must kill this sum.

Now assume $f_{\alpha_1 + \cdots + \alpha_{k-1}}\nu^+ \notin V_{r,k-1}$. Consider the summands in $(*)$ which give a multiple of $f_{\alpha_1 + \cdots + \alpha_{k-1}}\nu^+$ when some $e_\gamma_1 \cdots e_\gamma_l$ (where $\gamma_1 = \alpha_k + \cdots + \alpha_{k+l}, \gamma_2 = \alpha_{k+l+1} + \cdots$ and $\sum \gamma_l = \alpha_{k+1} + \cdots + \alpha_m$) is applied. They are the $F_i \nu^+ (0 \leq l \leq m - k)$, and

$$G_i \nu^+ = f_{\alpha_1 + \cdots + \alpha_{k-1}}f_{\alpha_{i+1} + \cdots + \alpha_k}f_{\alpha_{i+2} + \cdots + \alpha_m}\nu^+, \quad 0 \leq l < m - k$$

(note that we need not include $G_{m-k} = f_{\alpha_1 + \cdots + \alpha_{k-1}}f_{\alpha_{i+1} + \cdots + \alpha_m}\nu^+$ here by our assumption that $f_{\alpha_1 + \cdots + \alpha_m}\nu^+ \in V_{k,m}$).

By the assumption that $f_{\alpha_1 + \cdots + \alpha_{k-1}}\nu^+ \notin V_{r,k-1}$, it follows that $f_{\alpha_1 + \cdots + \alpha_{k-1}}\nu^+ \neq 0$. So any $e_\gamma_1 \cdots e_\gamma_l$ ($\gamma_l$ as in the last paragraph) must kill

$$\sum_{l=0}^{m-k} b_l F_i \nu^+ + \sum_{l=0}^{m-k-1} c_l G_i \nu^+,$$
where \( c_i \) is the coefficient of \( G_i \) in (\(*\)). In particular, \( e_{a_1}E_3 \) must kill \( \sum c_i G_i v^+ \) for every \( 0 \leq s < m - k \). The following shows that \( E_s G_i v^+ = 0 \) if \( s \neq 1 \). Assume \( s < l \). Then

\[
E_s G_i v^+ = (e_{a_{k_1}+\ldots+a_{k_s}}e_{a_{k_1+1}+\ldots+a_{k_s}}f_{a_{k_1}+\ldots+a_{k_s}}f_{a_{k_1+1}+\ldots+a_{k_s}}v^+) = 0
\]

for some structure constant \( d \). A similar calculation holds for \( s > l \). Also,

\[
e_{a_1}E_i G_i v^+ = e_{a_1}f_{a_1+a_2+a_3+\ldots+a_m}f_{a_1+a_2+a_3+\ldots+a_m}v^+ = \pm e_{a_1}f_{a_1+a_2+a_3+\ldots+a_m}f_{a_1+a_2+a_3+\ldots+a_m}v^+ = \pm a_1 a_m f_{a_1+a_2+a_3+\ldots+a_m}v^+.
\]

So:

\[
0 = e_{a_1}E_0 (\sum c_i G_i v^+) = \pm c_0 a_1 a_m f_{a_1+a_2+a_3+\ldots+a_m}v^+ \Rightarrow c_0 = 0
\]

\[
0 = e_{a_1}E_1 (\sum c_i G_i v^+) = \pm c_1 a_1 a_m f_{a_1+a_2+a_3+\ldots+a_m}v^+ \Rightarrow c_1 = 0
\]

\[
\vdots
\]

\[
0 = e_{a_1}E_{m-k-1} (\sum c_i G_i v^+) = \pm c_{m-k-1} a_k a_m f_{a_1+a_2+a_3+\ldots+a_m}v^+ \Rightarrow c_{m-k-1} = 0.
\]

So this implies \( \sum c_i G_i v^+ = 0 \). But then:

\[
0 = e_{a_1+a_2+a_3+\ldots+a_m} (\sum b_i F_i v^+ + \sum c_i G_i v^+) = e_{a_1+a_2+a_3+\ldots+a_m} (\sum b_i F_i v^+)
\]

with

\[
d = \begin{cases} 
\pm 2 & \text{if } m = n \\
\pm 1 & \text{otherwise}
\end{cases}
\]

(as \( F_{m-k} v^+ = f_{a_1+a_2+a_3+\ldots+a_m} v^+ \) is the only term in \( \sum b_i F_i v^+ \) not killed by \( e_{a_1+a_2+a_3+\ldots+a_m} \)). But this is a contradiction, since we have assumed \( p \neq 2 \).

So our assumption that there was an \( r \), \( i \leq r < k \), such that \( f_{a_1+a_2+a_3+\ldots+a_r} v^+ \notin V_{r,k} \) and \( f_{a_1+a_2+a_3+\ldots+a_r} v^+ \notin V_{r,k-1} \) must be false; i.e. for every \( r \), \( i \leq r < k \), either (i) \( f_{a_1+a_2+a_3+\ldots+a_r} v^+ \in V_{r,k} \), or (ii) \( f_{a_1+a_2+a_3+\ldots+a_r} v^+ \in V_{r,k-1} \). We want to show that in fact (i) holds always. There are several cases:

(a) If \( r < k - 1 \) and \( f_{a_1+a_2+a_3+\ldots+a_r} v^+ \in V_{r,k-1} \), we see that:

\[
f_{a_1+a_2+a_3+\ldots+a_r} v^+ = f_{a_1+a_2+a_3+\ldots+a_r} f_{a_1+a_2+a_3+\ldots+a_r} v^+ = \sum_{(p_1, \ldots, p_m) \geq 2} f_{p_1} f_{p_2} f_{p_3} \ldots f_{p_m} v^+ - f_{a_1+a_2+a_3+\ldots+a_r} f_{a_1+a_2+a_3+\ldots+a_r} v^+ = \sum_{(p_1, \ldots, p_m) \geq 2} f_{p_1} f_{p_2} f_{p_3} \ldots f_{p_m} v^+ - f_{a_1+a_2+a_3+\ldots+a_r} f_{a_1+a_2+a_3+\ldots+a_r} v^+
\]

for some integers \( s \) (depending on \( (p_1, \ldots, p_m) \)), so \( f_{a_1+a_2+a_3+\ldots+a_r} v^+ \in V_{r,k} \).

(b) If \( r = k - 1 \) and \( a_{k-1} \neq 0 \), (ii) cannot happen, since \( V_{k-1,k-1} = 0 \) and \( f_{a_1+a_2+a_3+\ldots+a_r} v^+ = f_{a_{k-1}} v^+ \neq 0 \).

(c) If \( r = k - 1 \) and \( a_{k-1} = 0 \), then

\[
f_{a_1+a_2+a_3+\ldots+a_r} v^+ = \pm (f_{a_1+a_2+a_3+\ldots+a_r} f_{a_1+a_2+a_3+\ldots+a_r} v^+) = \pm f_{a_{k-1}} f_{a_{k-1}} v^+,
\]

since \( a_{k-1} = 0 \). So \( f_{a_1+a_2+a_3+\ldots+a_r} v^+ = f_{a_{k-1}} v^+ \in V_{k-1,k} \). So the Lemma is proved.

Since there are fewer non-zero labels between \( a_i \) and \( a_k \) than between \( a_i \) and \( a_m \), by our inductive hypothesis \( f_{a_1+a_2+a_3+\ldots+a_r} v^+ \in V_{r,j} \). So the Proposition is proved.

\[\square\]
3.2. The case $D_n < B_n$. Assume $X = D_n$ is embedded in $Y = B_n$ in the usual way (as the derived group of the stabilizer of a 1-space in the natural module for $Y$). Now $V$ is a restricted irreducible $KB_n$-module, with high weight $\lambda = a_1\lambda_1 + a_2\lambda_2 + \cdots + a_n\lambda_n$ (recall that $\lambda_i$ is the fundamental dominant weight corresponding to the root $\alpha_i$ of $Y = B_n$) and high weight vector $v^+$. Then $D_n$ is the subgroup of $B_n$ generated by all the root subgroups corresponding to long roots, and $t$ may be chosen to be a representative in $B_n$ of the Weyl group reflection $x_{\alpha_0}$. The $K$-vector space $V$ is irreducible as a $KD_n(t)$-module, but not as a $DB_n$-module. The high weights of $V$ as a $D_n$-module are restricted. The symbol “$\equiv$” will mean congruent modulo $p$.

**Theorem 3.3.** If $p \not\equiv 2$, then $V$ restricted to $KD_n(t)$ is irreducible if and only if $a_i = 1$, $(\lambda + p, \alpha_i + \cdots + \alpha_j) \equiv 1$ for every pair of successive non-zero coefficients $a_i, a_j$ with $i < j \leq n$. If $p = 2$, then $V$ restricted to $KD_n(t)$ is irreducible if and only if $a_i = 1$ and $a_i = 0$ for $i < n$.

Note that in any characteristic, the spin module (high weight $\lambda_n$) for $B_n$ is the sum of two spin modules for $D_n$.

**Proof.** We will work over the Lie algebras of $X$ and $Y$, rather than the group algebras. By Lemma 1.1 ii) of [10], $V$ is an irreducible $DB_n$-module. The notation for elements of the Lie algebra was introduced in the previous section. The proof will consist of a series of Lemmas.

The following are the roots which correspond to nodes of the Dynkin diagrams for $X$ and $Y$ (since we know the embedding $X \rightarrow Y$ precisely, and $X$ contains the full maximal torus of $Y$, we know what the $\beta_i$ are in terms of the $\alpha_i$):

![Dynkin diagram](image)

The fundamental dominant weights for $X$ are

$$\lambda_1, \lambda_2, \ldots, \lambda_n$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the fundamental dominant weights for $B_n$.

**Lemma 3.4.** Assume that $a_n \leq 1$ and $p \not\equiv 2$. Then $V = L(D_n)v^+ \oplus L(D_n)f_{\alpha_n}v^+$ as a $KD_n$-module if and only if $f_{\alpha_1 + \cdots + \alpha_i}v^+ \in L(D_n)\alpha_i v^+$ and $f_{\alpha_1 + \cdots + \alpha_i}f_{\alpha_n}v^+ \in L(D_n)v^+$ for all $i < n$.

**Proof.** Note that $f_{\alpha_n}v^+$ is a maximal vector for $D_n$, and as mentioned above $D_n$ contains the full maximal torus of $B_n$.

$\Rightarrow$: First: $f_{\alpha_1 + \cdots + \alpha_i}v^+ \in L(D_n)f_{\alpha_n}v^+$. This is clear if $f_{\alpha_1 + \cdots + \alpha_i}v^+ = 0$ for every $i$. If this is not the case, let $i < n$ such that $0 \neq w = f_{\alpha_1 + \cdots + \alpha_i}v^+$. Then $w$ is a weight vector of weight $\mu = \lambda - (\alpha_1 + \cdots + \alpha_n)$.

So if $w = w_1 + w_2$, with $w_1 \in L(D_n)\alpha_1 + \cdots + \alpha_i + \alpha_n$ and $w_2 \in L(D_n)f_{\alpha_n}v^+$, then $w_1, w_2$ are both also weight vectors of weight $\lambda - (\alpha_1 + \cdots + \alpha_n)$; so $w_1 = 0$, since all weights appearing in $L(D_n)v^+$ are of the form $\lambda - \beta$ for $\beta \in \langle \alpha_1, \ldots, \alpha_n - 1, \alpha_n - 1 + 2\alpha_n \rangle$. Thus $w = w_2 \in L(D_n)f_{\alpha_n}v^+$. Similar arguments show that $f_{\alpha_1 + \cdots + \alpha_i}f_{\alpha_n}v^+ \in L(D_n)v^+$.

$\Leftarrow$: Let $V_1 = L(D_n)v^+$ and $V_2 = L(D_n)f_{\alpha_n}v^+$. We claim that if $f_{\alpha_1 + \cdots + \alpha_i}v^+ \in V_1 \oplus V_2$ and $f_{\alpha_1 + \cdots + \alpha_i}f_{\alpha_n}v^+ \in V_1 \oplus V_2$ for all $i < n$, then $V = V_1 \oplus V_2$.

**Proof (claim).** a) First we prove that the hypotheses of the claim imply

$$f_{\gamma_1} \cdots f_{\gamma_m} f_{\alpha_n}v^+ \in V_1 \oplus V_2$$

whenever $a = 0$ or 1, $\gamma_1$ is a short root, and $\gamma_2, \ldots, \gamma_m$ are long. Assume this is not the case, and let $f_{\gamma_1} \cdots f_{\gamma_m} f_{\alpha_n}v^+$ be a counterexample with $m$ minimal. By the hypothesis, $m \geq 2$. Then

$$f_{\gamma_1} \cdots f_{\gamma_m} f_{\alpha_n}v^+ = f_{\gamma_1} f_{\gamma_2} \cdots f_{\gamma_m} f_{\alpha_n}v^+ + N(\gamma_1, \gamma_2)f_{\gamma_1+\gamma_2} f_{\gamma_3} \cdots f_{\gamma_m} f_{\alpha_n}v^+$$
and \( f_{\beta_1}f_{\beta_2} \cdots f_{\beta_n}f_{\alpha_n}v^+ \) (thus \( f_{\beta_1}f_{\beta_2}f_{\beta_3} \cdots f_{\beta_n}f_{\alpha_n}v^+ \) and \( f_{\beta_1}f_{\beta_2}f_{\beta_3} \cdots f_{\beta_n}f_{\alpha_n}v^+ \) are in \( V_1 \oplus V_2 \) by the minimality of \( m \). But this contradicts our assumption that \( f_{\beta_1} \cdots f_{\beta_n}f_{\alpha_n}v^+ \notin V_1 \oplus V_2 \). So a) is proved.

b) The \( Y \)-module \( V \) is spanned by elements of the form \( f_{\beta_1} \cdots f_{\beta_k}v^+ \), with the \( \beta_i \) roots of \( Y = B_n \). Assume the claim is false, and let \( f_{\beta_1} \cdots f_{\beta_m}v^+ \) be one of these spanning elements not in \( V_1 \oplus V_2 \), with \( m \) minimal. By minimality, \( f_{\beta_1} \cdots f_{\beta_m}v^+ = f_{\beta_0}(\sum a_{f_{\beta_1}} \cdots f_{\beta_m}v^+ + \sum b_{f_{\beta_1}} \cdots f_{\beta_m}f_{\alpha_n}v^+) \), where all the \( \gamma_j, \epsilon_j \) are long roots. By a), and our assumption that \( f_{\beta_1}v^+, f_{\beta_2}f_{\alpha_1}v^+ \in V_1 \oplus V_2 \) if \( s = 0 \) or \( t = 0 \), \( f_{\beta_1} \cdots f_{\beta_m}v^+ \in V_1 \oplus V_2 \) (since the only short roots are the \( \alpha_1 + \cdots + \alpha_n \)), contrary to our choice of \( f_{\beta_1} \cdots f_{\beta_m}v^+ \). So the claim is proved.

Finally, it is clear that \( f_{\alpha_1, \ldots, \alpha_n}v^+ \in L(D_n)v^+ \oplus L(D_n)f_{\alpha_n}v^+ \) if and only if \( f_{\alpha_1, \ldots, \alpha_n}v^+ \in L(D_n)f_{\alpha_n}v^+ \), by considering the roots that appear in each summand; a similar argument holds for \( f_{\alpha_1, \ldots, \alpha_n}f_{\alpha_n}v^+ \). Putting this together with the claim, we have the lemma. 

Assume now that \( V \) is the direct sum of two irreducible \( L(D_n) \)-modules. The group \( X = D_n \) contains the full torus \( T \) of \( B_n \), so a \( B_n \)-high weight vector \( v^+ \in V \) must also be a weight vector for \( D_n \). If \( v^+ = v_1 + v_2 \) \((v_j \in V_j)\), then \( v_1, v_2 \) are both weight vectors of weight \( \lambda \), the \( B_n \)-high weight. But the high weight space in \( V \) has dimension 1. So \( v_1 = 0 \) or \( v_2 = 0 \), which implies that \( v^+ \) is in one of the summands \( V_1, V_2 \). Fix a Borel subgroup \( B' \) of \( D_n \) contained in the Borel subgroup \( B \) of \( B_n \). Since \( B \) stabilizes \( K v^+ \), so does \( B' \). Thus \( v^+ \) is a maximal vector in one of the irreducible summands. So one of the summands has marking

\[
\begin{array}{cccccccc}
  a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} & a_n \\
  \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \downarrow \\
  a_{n-1} + a_n
\end{array}
\]

Call the \( D_n \)-module with this marking \( V_1 \). Since \( t \) interchanges the two summands and corresponds to the graph automorphism, the other summand has marking

\[
\begin{array}{cccccccc}
  a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} + a_n \\
  \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow \\
  a_n
\end{array}
\]

We call this \( D_n \)-module \( V_2 \). Notice another expression for this high weight: \( a_1 \lambda_1 + \cdots + a_{n-2} \lambda_{n-2} + (a_{n-1} + a_n)(\lambda_{n-1} - \lambda_n) + a_{n-1} \lambda_n = a_1 \lambda_1 + \cdots + a_{n-2} \lambda_{n-2} + (a_{n-1} + a_n) \lambda_{n-1} - a_n \lambda_n = \lambda - a_n \alpha_n \).

Since the \( B_n \)-high weight space has dimension 1, \( a_n \neq 0 \) (else there is a \( B_n \)-high weight vector in each summand, which is impossible as above). In the second summand, the high weight is \( a_1 \lambda_1 + \cdots + a_{n-2} \lambda_{n-2} + (a_{n-1} + a_n) \lambda_{n-1} - a_n \lambda_n = \lambda - a_n \alpha_n \).

The weight \( \lambda - \alpha_n \) appears in \( V \) (since \( a_n \neq 0 \)). So it must appear in one of the \( D_n \)-summands (the earlier argument that \( v^+ \) is in one of the summands shows that any weight which appears in \( V \) must appear in at least one of the summands). It clearly cannot appear in \( V_1 \) (the only weights appearing there are \( \lambda - \beta \) for \( \beta \in \{ \alpha_1, \ldots, \alpha_{n-1}, 2 \alpha_n \} \)), so it must be in \( V_2 \), which implies \( a_n = 1 \).

Note that \( V_2 = L(D_n)f_{\alpha_n}v^+ \) and \( V_1 = L(D_n)v^+ \). So by the last Lemma,

\[
f_{\alpha_1, \ldots, \alpha_n}v^+ \in L(D_n)f_{\alpha_n}v^+
\]

for all \( 1 \leq i \leq n \).

For \( w \in V \), let \( L(D_n)w \) denote the span of all \( f_{\beta_1} \cdots f_{\beta_m}w \) for \( \beta_i \in \Sigma^+(D_n) \). Notice that for \( w = v^+ \) or \( w = f_{\alpha_n}v^+ \), we have \( L(D_n)w = L(D_n)w \), as the \( e_{\beta} \) for \( \beta \in \Sigma^+(D_n) \) annihilate these vectors.

**Lemma 3.5.** Let \( V_{i,k} \) be as before. Fix \( i \) and \( m \) such that \( 1 \leq i < m \leq n \). Then \( f_{\alpha_1, \ldots, \alpha_m}v^+ \in V_{i,m} \) for all \( r \in [i,m] \) if and only if \( f_{\alpha_1, \ldots, \alpha_m}v^+ \in L(D_n)v^+ \) for all \( r \in [i,m] \).

**Proof.** \( \Leftarrow \): This is clear, as \( L(D_n)v^+ \cap V_{i,(a_1, \ldots, \alpha_n)} \subseteq V_{i,m} \).

\( \Rightarrow \): This is for \( m = n + 1 \), so assume \( s = m - i > 1 \) and we have proved the Lemma for \( m - i < s \) and that \( f_{\alpha_1, \ldots, \alpha_n}v^+ \in V_{i,m} \) for all \( i \leq r < m \). Consider \( r \) such that \( i \leq r < m \). If \( i < r \) then by the inductive hypothesis \( f_{\alpha_1, \ldots, \alpha_n}v^+ \in L(D_n)v^+ \), and \( f_{\alpha_1, \ldots, \alpha_n}v^+ \) is a linear combination of terms of the
form \(f_{a_1 + \cdots + a_r} v^+\), where \(r > i\). All these terms are in \(L(D_n)^- f_{a_m} v^+\) by induction; thus so is \(f_{a_1 + \cdots + a_m} v^+\).

Now by Proposition 3.1 and the last two Lemmas, \(f_{a_1 + \cdots + a_j} v^+ \in V_{i,j}\) whenever \(a_i, a_j\) are consecutive non-zero labels: Lemma 3.4 implies that \(f_{a_1 + \cdots + a_i} v^+ \in L(D_n) f_{a_m} v^+\) for every \(i\); then Lemma 3.5 gives \(f_{a_1 + \cdots + a_k} v^+ \in V_{r,n}\) for every \(i\) and every \(i \leq r < n\); finally, Proposition 3.1 implies that \(f_{a_1 + \cdots + a_j} v^+ \in V_{i,j}\) for every \(i\), where \(a_j\) is the first non-zero coefficient after \(a_i\). Recall that \(a_n = 1\), and that “\(\equiv\)” means congruent modulo \(p\).

**Lemma 3.6.** Let \(1 \leq i < j \leq n\) such that \(a_i = 0\) for \(i < l < j\), but \(a_i \neq 0 \neq a_j\). Then

\[
f_{a_1 + \cdots + a_j} v^+ \in V_{i,j} \iff \begin{cases} 2a_i \equiv -2(j-i)-1 & \text{if } j = n \\ a_i + a_j \equiv i - j & \text{if } j < n. \end{cases}
\]

**Proof.** The \(\lambda - (\alpha_i + \cdots + \alpha_j)\)-weight space is spanned by

\[
\{ f_{k} v^+ = f_{a_1 + \cdots + a_{i-1}, a_i, f_{a_{j+1} + \cdots + a_j}} v^+ | 1 \leq k < j \} \cup \{ f_{a_1 + \cdots + a_j} v^+ \}.
\]

So \(f_{a_1 + \cdots + a_j} v^+ \in V_{i,j}\) if and only if there is a relation

\[
0 = f_{a_1 + \cdots + a_j} v^+ + \sum_{k=i}^{j-1} b_k f_{k} v^+.
\]

(The vector on the right hand side of \((***)\) is 0 if and only if it is killed by all \(e_{\alpha_i}\):)

For \(l < i \) or \(i > j\), \(e_{\alpha_i}(f_{a_1 + \cdots + a_j} v^+ + \sum_{k=i}^{j-1} b_k f_{k} v^+) = 0\) trivially.

So if \(j = n\), the above implies that \(b_1 = \cdots = b_{j-1} = -2a_j^{-1} = -2\). Then the last equation (for \(e_{\alpha_i}\)) gives the relation \(-1 - 2(a_i + 1) + (j-i-1)(-2) = 0\); i.e. \(-2a_i - 2(j-i) - 1 = 0\), or \(2a_i = -2(j-i) - 1\).

If \(j < n\), then \(b_1 = \cdots = b_{j-1} = -\frac{1}{a_j}\). So the equation for \(e_{\alpha_i}\) gives \(-1 - \frac{1}{a_j}(a_i + 1) - \frac{1}{a_j}(j-i-1) = 0\), or \(a_j + a_i = i - j\).

This completes the proof of the Lemma.

That the set of congruences given in the Lemma is equivalent to the conclusion of Theorem 3.3 is clear: If \(a_k\) is the last non-zero label before \(a_n\), then

\[
\begin{align*}
1 & \equiv (\lambda + \rho, \alpha_k + \cdots + \alpha_n) = 2(\lambda + \rho, \alpha_k + \cdots + \alpha_{n-1}) + (\lambda + \rho, \alpha_n) = 2(a_k + 1) + 2(n-k-1) + 2 \\
\iff 2a_k & \equiv -2(n-k) - 1.
\end{align*}
\]
Also, if $a_i, a_j$ are consecutive non-zero labels with $j < n$, then
\[ 1 \equiv (\lambda + \rho, \alpha_i + \cdots + \alpha_j) = (\lambda + \rho, \alpha_i + \cdots + \alpha_j) = a_i + a_j + (j - i + 1) \]
\[ \iff a_i + a_j \equiv i - j. \]

This completes the proof of Theorem 3.3 in one direction.

Now assume $V$ is as in the setup of the theorem (an irreducible restricted $KB_{rs}$-module), and that $\lambda$ satisfies the congruences in the last Lemma (by the comments above, this is equivalent to satisfying the congruences in the Theorem). We need to show $V$ is irreducible as a $(D_n, \tau)$-module.

By Lemmas 3.4 and 3.5, if we show $f_{\alpha_1 + \cdots + \alpha_n} v^+ \in V_{i,n}$ and $f_{\alpha_1 + \cdots + \alpha_n} f_{\alpha_i} v^+ \in L(D_n) v^+$ for all $i < n$, then $V = L(D_n) v^+ \oplus L(D_n) f_{\alpha_i} v^+$ as an $L(D_n)$-module. First we show that in fact $f_{\alpha_1 + \cdots + \alpha_n} v^+ \in V_{i,m}$ for every $i$ and $m$ with $1 \leq i < m \leq n$:

a) Assume there are no non-zero labels between $a_i$ and $a_m$. Then by Lemma 3.6, $f_{\alpha_1 + \cdots + \alpha_m} v^+ \in V_{i,m}$, there is a relation
\[ f_{\alpha_1 + \cdots + \alpha_n} v^+ = -\sum_{k=1}^{m-1} b_i f_{\alpha_i} v^+ \in L(D_n) f_{\alpha_i} v^+ \subseteq V_{i,n}. \]

b) Assume there are $r > 2$ non-zero labels between $a_i$ and $a_m$ (inclusive), and that we have proved the result for fewer than $r$ non-zero labels. Then
\[ f_{\alpha_1 + \cdots + \alpha_n} v^+ = \pm (f_{\alpha_1 + \cdots + \alpha_{m-1}} f_{\alpha_m} v^+ - f_{\alpha_m} f_{\alpha_1 + \cdots + \alpha_{m-1}} v^+). \]

We have $f_{\alpha_1 + \cdots + \alpha_{m-1}} f_{\alpha_m} v^+ \in V_{i,m}$, and $f_{\alpha_1 + \cdots + \alpha_{m-1}} v^+ \in V_{i,m-1}$ by induction. So $f_{\alpha_1 + \cdots + \alpha_{m-1}} v^+$ is a sum of terms of type (1) with more than one $f_\alpha$. Then $f_{\alpha_i}$ commutes with all but the last $f_\beta$ in each of the terms; thus $f_{\alpha_i} f_{\alpha_1 + \cdots + \alpha_{m-1}} v^+ \in V_{i,m}$. So $f_{\alpha_1 + \cdots + \alpha_n} v^+ \in V_{i,m}$.

Next we must show that $f_{\alpha_1 + \cdots + \alpha_n} f_{\alpha_i} v^+ \in L(D_n) v^+$ for every $i < n$. We use induction, doing the base cases much as in the proof of Lemma 3.6.

Let $i = n - 1$. The $\lambda - (\lambda_{n-1} + 2\alpha_n)$-weight space is spanned by
\[ \{ f_{\alpha_1 + \cdots + \alpha_{n-1} + 2\alpha_n} v^+, f_{\alpha_1 + \cdots + \alpha_{n-1}} f_{\alpha_n} v^+, f_{\alpha_1 + \cdots + \alpha_{n-1}} f_{\alpha_n} f_{\alpha_n} v^+ \}. \]

The third element in this set is 0, as $a_n = 1$. Set $b = 1$ if $a_{n-1} = 0$, and set $b = -2$ if $a_{n-1} \neq 0$. Then the vector
\[ w = f_{\alpha_1 + \cdots + \alpha_{n-1}} f_{\alpha_n} v^+ + b f_{\alpha_1 + \cdots + \alpha_{n-1}} 2\alpha_n v^+ \]
is trivially annihilated by $e_{\alpha_l}$ for $1 < l < n - 1$; each summand is annihilated by $e_{\alpha_{n-1}}$; and, finally,
\[ e_{\alpha_l} f_{\alpha_1 + \cdots + \alpha_{n-1}} f_{\alpha_n} v^+ + b e_{\alpha_l} f_{\alpha_1 + \cdots + \alpha_{n-1} + 2\alpha_n} v^+ = f_{\alpha_1 + \cdots + \alpha_{n-1} + \alpha_l} v^+ + 2 f_{\alpha_1 + \cdots + \alpha_{n-1}} f_{\alpha_n} v^+ + b f_{\alpha_1 + \cdots + \alpha_{n-1} + \alpha_l} v^+ = \begin{cases} (1 + 2N(-m_{n-1} - n) + b) f_{\alpha_1 + \cdots + \alpha_{n-1} + \alpha_l} v^+ = (b - 1) f_{\alpha_1 + \cdots + \alpha_{n-1} + \alpha_l} v^+ = 0 & \text{if } a_{n-1} = 0, \\ (b + 2) f_{\alpha_1 + \cdots + \alpha_{n-1} + \alpha_l} v^+ = 0 & \text{if } a_{n-1} \neq 0. \end{cases} \]

Since $w$ is not a high weight vector, this shows that it must in fact be 0, so $f_{\alpha_1 + \cdots + \alpha_{n-1}} f_{\alpha_n} v^+ \in L(D_n) v^+$.

Next let $i = n - 2$. Now the $\lambda - (\lambda_{n-2} + \lambda_{n-1} + 2\alpha_n)$-weight space is spanned by
\[ \{ f_{\alpha_1 + \cdots + \alpha_{n-2} + 2\alpha_n} v^+, f_{\alpha_1 + \cdots + \alpha_{n-2} + \alpha_1} f_{\alpha_n} v^+, f_{\alpha_1 + \cdots + \alpha_{n-2} + 2\alpha_n} f_{\alpha_1 + \cdots + \alpha_{n-2} + 2\alpha_n} v^+ \}, \]
as we just showed that we needn’t include $f_{\alpha_1 + \cdots + \alpha_{n-2} + \alpha_1} f_{\alpha_n} f_{\alpha_n} v^+ + f_{\alpha_1 + \cdots + \alpha_{n-2} + 2\alpha_n} f_{\alpha_1 + \cdots + \alpha_{n-2} + 2\alpha_n} v^+$, since $a_n = 1$.

If $a_{n-1} = 0$, let $b_1 = 2$ and $b_2 = -2$. If $a_{n-1} \neq 0$, let $b_1 = 2$ and $b_2 = 1$. Then in calculations similar to those above, we find that $e_{\alpha_l}$ annihilates
\[ w = f_{\alpha_1 + \cdots + \alpha_{n-2} + \alpha_1} f_{\alpha_n} v^+ + b_1 f_{\alpha_1 + \cdots + \alpha_{n-2} + \alpha_1} f_{\alpha_n} v^+ + b_2 f_{\alpha_1 + \cdots + \alpha_{n-2} + \alpha_1} f_{\alpha_n} v^+ \]
so in fact this sum is 0 and $f_{\alpha_1 + \cdots + \alpha_{n-2} + \alpha_1} f_{\alpha_n} f_{\alpha_n} v^+ \in L(D_n) v^+$ (the only complication is that in the case $a_{n-1} \neq 0$, we were forced to again apply the $e_{\alpha_l}$ to show that $e_{\alpha_l} w = 0$).

Finally, assume that $i \leq n - 3$. Then
\[ f_{\alpha_1 + \cdots + \alpha_i} f_{\alpha_i} v^+ = f_{\alpha_1 + \cdots + \alpha_i} f_{\alpha_i} v^+ + b f_{\alpha_1 + \cdots + \alpha_i} 2\alpha_i v^+ \]
for some $b$, so $f_{\alpha_1 + \cdots + \alpha_n} \phi \cdot v^+ \in L(D_n) v^+$ if and only if $f_{\alpha_i} \phi \cdot v^+ \in L(D_n) v^+$. We have

\[
\begin{align*}
\phi \cdot v^+ &= \pm (f_{\alpha_1} \phi \cdot v^+ - f_{\alpha_2} \phi \cdot v^+) \\
&= \pm (f_{\alpha_1} \phi \cdot v^+ - f_{\alpha_2} \phi \cdot v^+) \\
&= \pm (f_{\alpha_1} \phi \cdot v^+ - f_{\alpha_2} \phi \cdot v^+) \\
&= \pm (f_{\alpha_1} \phi \cdot v^+ - f_{\alpha_2} \phi \cdot v^+)
\end{align*}
\]

where $w \in V_{i,n-2} (\text{see above})$ and $w$ is a sum of terms of type (i) with at least two $f_{\alpha_i}$. Thus $f_{\alpha_1} \phi \cdot v^+$ commutes with all but the last of these $f_{\alpha_i}$, and by induction we are again in $L(D_n) v^+$. So $f_{\alpha_1 + \cdots + \alpha_n} \phi \cdot v^+ \in L(D_n) v^+$ for every $i < n$.

We still must show that $V_1 = L(D_n) v^+$ and $V_2 = L(D_n) f_{\alpha_i} v^+$ are irreducible $L(D_n)$-modules. By the above, $V = V_1 \oplus V_2$ as $L(D_n)$-modules. Also, $V \cong V^*$ as $L(B_2)$-modules, thus as $L(D_n)$-modules. Now consider $V$ as an $L(D_n)$-module; it has a quotient $V/\lambda_i$. If $\lambda$ is even, then $V/\lambda_i \cong V/\lambda_j$ (by equivalence of Dynkin diagrams), so $V^+ \cong V$ has $V/\lambda_i$ as a submodule. Since $\lambda$ is the only vector of weight $\lambda$, and $L(D_n) v^+ = V_1$, we must have $V_1 \cong V/\lambda_i$. If $\lambda$ is odd, then $V/\lambda_i \equiv V/\lambda_j$ as a submodule. Since $f_{\alpha_1} v^+$ spans $V_{\lambda-\alpha_i}$, we have $V_2 = L(D_n) f_{\alpha_i} v^+ \cong V(\lambda_j - \alpha_i)$. Now doing the same thing with $V(\lambda_j - \alpha_i)$ instead of $V/\lambda_i$, we see that $V_1 \cong V/\lambda_i$ and $V_2 \cong V(\lambda_j - \alpha_i)$. So $V_1, V_2$ are irreducible $L(D_n)$-modules.

So the Theorem is proved for $p = 2$.

Now assume $p = 2$. By [10, 6.1], $V = V' \otimes V''$ as a $KR_{2p}$ module (thus as a $KD_{2p}$ module), where $V'$ is the irreducible module with highest weight $\lambda_1 \lambda_1 + \cdots + \lambda_{n-1} \lambda_{n-1}$, and $V''$ has highest weight $\alpha_n$. If we assume that $V$ is the sum of two irreducibles for $L(D_n)$ ($V = V_1 \oplus V_2$), then the same argument as before shows $\alpha_n = 1$. Then $V''$ is the sum of two non-zero irreducibles for $L(D_n)$, say $V'' = W_1 \oplus W_2$. So $V' \otimes V'' = (V' \otimes W_1) \oplus (V' \otimes W_2) \cong V_1 \oplus V_2$ as $D_n$-modules. But the same result in [10] says that no restricted irreducible $D_n$-modules are tensor decomposable. So one of $V', V''$ must be 0. Since $V'' \neq 0$, we must have $V' = 0$, so $a_1 = a_2 = \cdots = a_{n-1} = 0$.

This completes the proof of Theorem 3.3.

\[\square\]

4. When \( W | X \) is reducible

The notation will be as in the introduction: $G$ is an algebraic group over a field $K$ of arbitrary characteristic $p$ (0 or a prime), with simple identity component $X$ admitting an outer (graph) automorphism. So except for the case $X = D_4$, the Dynkin diagram for $X$ has a single graph automorphism inducing an automorphism $t$ of $X$, and $G = X(t)$. If $X = D_4$, then $\text{Aut}(X) = D_4 \cdot \text{Sym}_3$. Let $\text{Sym}_3 = \langle \delta, t \rangle$, with $t^2 = 1$. Then the possibilities are: $G = D_4(t)$; $G = D_4(\delta)$; and $G = D_4(\delta, t)$.

Let $Y$ be a simple algebraic group of classical type and of rank $n$ such that $X \leq Y$ and $G \leq \text{Aut}(Y)$ (it may or may not be in $Y$; but if $X = D_4$ and $\delta \in G$ then $\delta Y \in X$ since no simple group properly containing $D_4$ has an outer automorphism of order 2). Let $\{\lambda_i\}$ be the set of fundamental dominant weights of $Y$ (X) and $\{\alpha_i\}$ the set of fundamental roots for $Y$ (X) with respect to some maximal torus $T_Y \subset Y$ (maximal torus). Let $V(\lambda)$ be an irreducible $Y$-module on which $G$ acts irreducibly but $X$ acts reducibly, with $\lambda = \sum a_i \lambda_i$.

In this section we consider the case when $X$ acts reducibly on the natural module $W$ for $Y$. The main theorem of the section is the following:

**Theorem 4.1.** Assume $X$ acts reducibly on the natural module $W$ for $Y$. Then either $Y = \text{SL}(V)$, $\text{SO}(V)$, $\text{Sp}(V)$, or $(X, Y, V)$ are as in $U_1$, $U_2$, $U_3$, or $U_4$ of Table 2.

The approach will be to analyze the various possibilities for a minimal $X$- or $G$-invariant subspace of $W$.

### 4.1. Preliminary Lemmas

Several possible situations for the action of $X$ on $W$ will arise more than once, so we have some Lemmas to prove first.

**Lemma 4.2.** Assume $P_Y$ is a maximal parabolic subgroup of $Y$ corresponding to the root $\alpha_j$, with $j = n$ if $Y$ is not of type $A_n$. Let $P_Y = L_Y Q_Y$ be a Levi decomposition. If there exist only two non-zero quotients $V^i(Q_Y) = [V_i Q_Y^{-1}] / [V_i Q_Y^0]$, and $\dim(V^i(Q_Y)) = \dim(V^2(Q_Y))$, then one of the following holds:
Proof. Let $\Lambda$ be the set of $T_i$-weights of $V$. Write $V^i = \sum_{\mu \in \Lambda_i} V_\mu$, where $\Lambda_i = \{\mu \in \Lambda | \mu = \lambda - \beta, \text{coefficient of } \alpha_j \text{ in } \beta = i - 1\}$. By 2.7 in [10], $V^1 \cong V/[V, Q_T] = V^1(Q_T)$ and $V^2 \cong [V, Q_T]/[V, Q_T, Q_T] = V^2(Q_T)$, so $\dim(V^1) = \dim(V^2)$.

Since $V = V^1 \oplus V^2$, the weight space $V_{w_0\lambda}$ must be in $V^2$ (if $V_{w_0\lambda} \subseteq V^1$ then $V^2 = 0$, so $V = V^1 = V/[V, Q_T]$ — thus $V = V^1$ is an irreducible $L_1$-module, which is a contradiction as $L_1 \subset P_1$, i.e., $w_0\lambda = \lambda - (\lambda_1 + \lambda_2 + \cdots)$). So $\lambda - w_0\lambda$ has $\alpha_j$-coefficient 1. In the cases where $w_0 = -1$ ($D_n$ for $n$ even, $B_n, C_n$; here $j = n$), this implies $2\lambda = \cdots + \alpha_n$. In the other cases ($A_n, D_n$ for $n$ odd), let $\sigma$ be an involutory graph automorphism of the group; then the above gives $\lambda + \lambda^\sigma = \cdots + \alpha_j + \cdots$ (with $j = n$ in the $D_n$ case). The cases are:

- $Y = A_n$, $L'_j = A_{j-1} \times A_{n-j}$: We use the expression for the $\lambda_i$ in terms of the $\alpha_i$ (e.g., [7, page 69]) to calculate the coefficient (which must be 1 by the above) of $\alpha_j$ in $\lambda + \lambda^\sigma$. By reversing the labelling if necessary, we may assume $j \leq (n+1)/2$. It is a straightforward calculation to see that we obtain $1 = (a_1 + a_n) + 2(a_2 + a_{n-1}) + \cdots + j(a_j + a_{j+1} + \cdots + a_{n-j+1})$. The only nonnegative integral solutions to this equation are $a_1 + a_n = 1, a_i = 0$ for $i \neq n$; and $j = 1, \sum a_i = 1$. The first gives $V \cong W$. So assume $j = 1, i = \sum a_i$. Assume $a_k = 1, a_i = 0$ for $i \neq k$. By assumption, $\dim(V/[V, Q_T]) = \dim([V, Q_T]/[V, Q_T, Q_T])$.

By Lemma 2.2, the high weight of $V/[V, Q_T]$ as an $L'_1$-module is $\sum_{i=2}^n a_i\lambda_i = \lambda_k|_{L'_1}$ ($k = 1$ cannot occur because this would give a top quotient of dimension 1 (high weight 0) and $\lambda_2$ as a high weight for the bottom quotient (as $L'_1$-modules) by Lemma 2.4; but we know the two quotients have the same dimension). The bottom quotient has a high weight $\lambda - (\alpha_1 + \cdots + \alpha_k)|_{L'_1} = \lambda_{k+1}|_{L'_1}$, and since for any $k$ the dimension of the $Y$-module with high weight $\lambda_k$ ($\binom{n}{k}$) is the sum of the dimensions of the $L'_j$-modules with high weights $\lambda_k|_{L'_j}$ and $\lambda_{k+1}|_{L'_j}$ ($\binom{n}{k} + \binom{n}{k+1}$), $\lambda_{k+1}$ is in fact the only high weight of the bottom quotient. These two must have the same dimension, so $\binom{n-1}{k} = \binom{n}{k}$; thus $n$ is odd and $k = (n+1)/2$. So this setup does give an example of two modules for $A_{n-1}$ summing to a module for $A_n$, and we get item (2) of the Lemma.

- $Y = B_n$: Here the coefficient of $\alpha_n$ in $2\lambda$ is $2 \sum_{i=1}^{n-1} a_i, a_{n}n$. This must be 1, which is impossible.

- $Y = C_n$: The coefficient of $\alpha_n$ in $2\lambda$ is $\sum_{i=1}^{n-1} a_i, a_{i}$. This coefficient must be 1, which implies $a_1 = 1, a_i = 0$ for $i > 1$. So here we have only the case $V \cong W$.

- $Y = D_n$, $j$ odd: The coefficient of $\alpha_n$ in $\lambda - w_0\lambda = \lambda + \lambda^\sigma$ is

$$\sum_{i=1}^{n-2} a_i, a_{n-1} + a_n \binom{n-1}{2}.$$  

This coefficient must 1, which implies either $a_1 = 1, a_i = 0$ for $i > 0$, which gives $V \cong W$, or $n = 3, a_1 = 0, a_2 + a_3 = 1$. In the second instance, our assumption is that $\Pi(L'_1) = \{\alpha_1, \alpha_2\}$, so there are two distinct possibilities:

- $a_2 = 1$: In the chain $V > [V, Q_T] > 0$, the top factor is an irreducible $L'_j = A_2$-module with high weight $\lambda_2|_{L'_1}$; the second factor has high weight $\lambda - (\alpha_1 + \alpha_2 + \alpha_3)|_{L'_j} = 0$. These two have different dimensions, so we get no examples here.

- $a_3 = 1$: As above, the top factor has high weight 0 for $L'_j$; the second factor has high weight $\lambda_3 = \lambda_1|_{L'_1}$. Again, these do not have the same dimension.
• $Y = D_4, n \geq 4$ even: The coefficient of $\alpha_n$ in $2\lambda$ is
\[
\sum_{i=1}^{n-2} a_i i + a_{n-1} \left( \frac{n-2}{2} \right) + a_n \left( \frac{n}{2} \right).
\]
This must be 1, which implies either $a_1 = 1, a_i = 0$ for $i > 1$, which gives $V \cong W$; or $n = 4, \lambda = \lambda_3$, which gives $\text{dim}(V) = 8$. Since $X$ is proper in $Y = D_4$, we have $X \neq D_4$ and hence $G = X(\mathfrak{t})$. So as $X$-modules, $V = V_1 \oplus V_2$, with $V_1$ and $V_2$ irreducible $X$-modules, interchanged by $t$.

We have $\text{dim}(V_1) = 4$, and the only simple groups of rank 4 or less with an irreducible module of dimension 4 are $A_1, C_2$, and $A_3$. So $X$ is of type $A_3$ (not $A_1$ or $C_2$ because $X$ admits a graph automorphism), and $V_1$ has high weight $\delta_1$ or $\delta_3$.

Every $A_3$-module of dimension $\leq 6p$ is completely reducible by [8], so $W$ is completely reducible as an $X$-module. Let $\delta$ be a $T_X$-high weight of $W$. If $\delta$ is not restricted, then by Steinberg’s tensor product theorem ([13]), $V_{2k}(\delta) = W_1^{q_{2k}} \otimes \cdots \otimes W_n^{q_{2n}}$, where the $W_i$ are restricted irreducible $X$-modules and the $q_i$ are distinct powers of $p$. If there are two or more terms in this product, then $\text{dim} V_{2k}(\delta) > 8$ (as each $W_i$ has dimension at least 4), so there is in fact only one term, $W_1^{q_1}$. Then but, then, as in the proof of Lemma 2.6, $q_1 = 1$, as there is no twist in the action of $X$ on $V$ (which factors through the embedding of $X$ in $Y$).

We must have $\delta_2$ or both $\delta_1$ and $\delta_3$ as $X$-high weights of $W$ (since the only $A_3$-high weight modules of dimension 8 or less are those with high weight $0, \delta_1, \delta_2$, or $\delta_3$; $X$ acts nontrivially on $W$, ruling out 0; and $t$ does act on $W$, so if $\delta_1$ appears then so does $\delta_3$). In fact, both of these possibilities give the other examples listed in the Lemma: Let $X$ be the derived group of the Levi factor of the parabolic subgroup corresponding to $\{\alpha_2, \alpha_3, \alpha_4\} \subseteq \Pi(Y)$. Then $X$ is of type $D_3 \cong A_1$; $W$ restricts to $X$ as $V_{A_1}(\delta_1) \oplus 2V_{A_1}(0)$, and the $Y$-module $V = V_{A_1}(\lambda_3)$ restricts to $X$ as $V_{A_1}(\delta_1) \oplus V_{A_1}(\delta_3)$. Similarly, if we let $X$ be the derived group of the Levi factor of the parabolic subgroup corresponding to $\{\alpha_1, \alpha_2, \alpha_3\}$, then $W$ restricts to $X$ as $V_{A_1}(\delta_1) \oplus V_{A_1}(\delta_3)$, as does $V$. In this latter case $V \cong W$ as $X$-modules, but not as $Y$-modules.

Finally, in the two cases above there is an element $t \in Y$ acting as a graph automorphism on the specified $A_3 \leq Y$. Consider the first case: $X$ corresponds to the subsystem $\{\alpha_2, \alpha_3, \alpha_4\} \subseteq \Pi(Y)$. Here $V$ restricts to $X$ as $V_{A_1}(\delta_1) \oplus V_{A_1}(\delta_3) = U \oplus U^*$, where $U$ is the natural module for $X$. Let $\{u_1, u_2, u_3, u_4\}$ be a basis for $U \leq V$, and $\{u_1^*, \ldots, u_4^*\}$ a basis for $U^*$. Then by Witt’s theorem, there is an element of $O(V)$ sending $u_i$ to $u_i^*$ and $u_i^*$ to $u_i$ for every $i$; this element is in fact of determinant 1 (it has determinant 1 on the 2-space $\langle u_i^*, u_i \rangle$), so is in $\text{SO}(V) = Y$.

Similarly consider the second of the possible embeddings $X \hookrightarrow Y$: $X$ corresponds to the subsystem $\{\alpha_1, \alpha_2, \alpha_3\} \subseteq \Pi(Y)$. Here $W$ restricts to $X$ as the sum of the natural module and its dual, and we proceed as above to find $t \in Y$.

\[
\square
\]

Recall that $V|_X = V_1 \oplus \cdots \oplus V_k$, with $V_i$ irreducible ($k = 2$ except possibly when $X = D_4$).

**Lemma 4.3.** Assume that $W|_X = W_1 \oplus \cdots \oplus W_k$ where each $W_i$ is irreducible as an $X$-module; $\text{rad}(W_i) = 0$ (so $Y$ has type $B_n, C_n, \text{ or } D_n$); and $G$ permutes the $W_i$ transitively. If $X \not\cong \text{I}(W_i)_t \times \cdots \times \text{I}(W_i)_t < Y$ and $\text{I}(W_i)_t$ acts trivially on every $V|_X$ on which $\text{I}(W_i)_t$ (for some $i \neq j$) acts nontrivially, then $V \cong W$.

**Proof.** Let $Y_i = \text{I}(W_i)_t$. When we say that $Y_i$ acts trivially on $V_\mu$, we mean that $1 \times \cdots \times Y_i \times 1 \times \cdots \times 1$ acts as scalars on $V_\mu$. Since $G$ permutes the $Y_i$ transitively, $Y_i \cong Y_j$. The natural module $W$ is symplectic or orthogonal and $\text{rad}(W_i) = 0$, so each $Y_i$ is also symplectic or orthogonal. So the possibilities are $X \rightarrow Y_i$ irreducible on $V_i$ for every $i$, with $X$ proper in each $Y_i$; and $X \cong Y_i$ for every $i$. Examining Table 1 in [10], we see that there are no possibilities for the first option except perhaps $V_i$ = natural module for $Y_i$ for every $i$. But the natural modules for the $Y_i$ are the $W_i$ and they sum to the natural module for $Y$. So this gives $V \cong W$.

If $X \cong Y_i$, then since $X$ has a graph automorphism, we have $X \cong Y_i = D_m$ (not $A_n$ because $W_i$ is orthogonal or symplectic) for some $m; Y = D_m$ if $p = 2$; $Y = D_m$ or $C_m$ if $p = 2$.

Combine orthogonal bases $\{w^{(i)}_1, w^{(i)}_2, \ldots, w^{(i)}_m, w^{(i)}_{-m}\}$ for the $W_i$ to get an orthogonal basis for $W$. Write $W_i^+ = \langle w^{(i)}_1, \ldots, w^{(i)}_m \rangle$; then let $P_Y = \text{stab}(W_i^+) \times \text{stab}(W_2^+) \times \cdots \times \text{stab}(W_n^+) \times \text{stab}(W_l)$. With respect to an
appropriate Borel subgroup and maximal torus, $P_Y$ is the parabolic subgroup of $Y$ corresponding to $\Pi(Y) - \{\alpha_m, \alpha_{2m}, \ldots, \alpha_{(r-1)m}\}$; and each simple factor of $L'_Y$ ($P_Y = L'_Y Q_Y$) is contained in a distinct $Y_i$.

Recall that $\lambda = \sum a_i \alpha_i$ is the $T_i$-high weight for $V$. As $Y_i$ acts trivially where $Y_i$ acts nontrivially, we may conclude that only one factor of $L'_Y$ acts nontrivially on the highest weight space $V_\alpha$ of the irreducible $L'_Y$-module $V/V(Q_i)$. If $a_{ij} \neq 0$ for some $j < l$, then $\lambda - a_{ij} \alpha_i$ is an $L'_j$-high weight in $[V_i, Q_j]/[V_i, Q_j, Q_l]$, and has non-zero labels on two connected components of $\Pi(L'_j)$, so as above we can conclude that in fact $a_{ij} = 0$ for every $j < l$.

Let $j > m$ be minimal with respect to $a_j \neq 0$. By the above, $m \not\in j$ or $j = n$. Then $j = am + r$ for some $a$, with $0 \leq r < m$. If $r \neq m - 1$, then $\lambda - (a \alpha_m + \cdots + a_j)$ is a $L'_j$-high weight in $V^2(Q_Y)$ with non-zero labels on two connected components, which is a contradiction. If $r = m - 1$ and $j \neq n - 1$, then $\lambda - (a_j + \cdots + a_{(r+1)m})$ is the required high weight of $V^2(Q_Y)$. Finally, if $j = n - 1$ or $j = n$, then $\lambda - (a_{(r-1)m} + \cdots + a_j)$ (if $Y \cong C_n$) or $\lambda - (a_{(r-1)m} + \cdots + a_{(m-1)j})$ is a high weight giving the same contradiction. So $a_j = 0$ for $j > m$.

If $a_i \neq 0$ for some $1 < i < m$ or $a_1 > 1$ (let $i = 1$ in this case), with $a_j = 0$ for $j > i$, then $\lambda - (\lambda_i + \cdots + \lambda_m)$ is an $L'_j$-high weight in $[V_i, Q_l]/[V_i, Q_l, Q_j]$, again with non-zero labels on two connected components of $\Pi(L'_j)$. So the only possibility is $a_1 = 1, a_i = 0$ for $i > 1$; in other words, $V \cong W$.

**Lemma 4.4.** Assume that $G = X(t)$, and that $W = W_1 \oplus W_2$, with $X$ stabilizing $W_1$ and $W_2$, $t$ stabilizing the decomposition, and $\text{rad}(W_1) = 0$. Then one of the following holds:

1. $V \cong W$;
2. $U_2$ of Table 2; or
3. $U_3$ of Table 2.

**Proof.** Let $Y_1 = I(W_1)'$; $Y_2 = I(W_2)'$. Then $X \not\cong Y_1 \times Y_2$, with $t$ either interchanging the two factors or stabilizing each.

The cases are: A. $V$ is irreducible for $Y_1 \times Y_2$; B. $V$ is reducible for $Y_1 \times Y_2$.

A) We can read off the possibilities for $V$ and $Y_1 \times Y_2$ from Table 1 in [10]:

1. $\text{IV}_2, \text{IV}_3$, $S_6$ in [10]: $Y = D_{n+1}, \lambda = \lambda_n$ or $\lambda_{n+1}, Y_1 \times Y_2 = B_{n-k} \times B_k, V = V(\lambda), V|_{Y_1 \times Y_2} = V(\lambda_n) \otimes V(\lambda_k), \{\lambda_n\}$ and $\{\lambda_k\}$ are sets of fundamental dominant weights for $Y_1$ and $Y_2$; OR

2. $\text{MR}_5$ in [10]: $p = 2, \lambda = \theta, Y_1 \times Y_2 = Y_1 \times Y_2 = V(\lambda), V|_{Y_1 \times Y_2} = V(\lambda) \otimes V(\lambda), (\lambda, \lambda')$ as above.

In both these cases $V = V' \otimes V'' = V_1 \otimes V_2$ as $X$-modules, for some $X$-modules $V'$ and $V''$. If $V'$ or $V''$ is irreducible as an $X$-module, we can check Table 1 in [10] and see that there are no simple connected proper subgroups of $B_{n-k}$ (or $B_k$) which act irreducibly on $V'$ ($V''$). This would force $X \cong B_{n-k}$ (or $B_k$), which is impossible because $X$ admits a graph automorphism. So $V', V''$ are both reducible as $X$-modules; let $V'_1 \not\cong V'$ and $V''_1 \not\cong V''_1$ be proper non-zero $X$-invariant subspaces. Then $V'_1 \otimes V'_1, V''_1 \otimes V''_1$, and $V' \otimes V''$ are all proper non-zero $X$-invariant subspaces of $V$. But there are only two such $X$-invariant subspaces. So we have no examples here.

B) $V$ is reducible for $Y_1 \times Y_2$. Recall that $t$ normalizes $Y_1 \times Y_2$.

In Theorem 3.3 we saw that $V = V(\lambda)$ is irreducible for $D_n(t) < B_n$ if and only if $a = 1$ and $b = 0$, or $a = 1$ and $2b + 2n - 2n - 1 \equiv 0$ (mod $p$). But $2b + 2n - 2n - 1 \equiv 0$ imply that $1 \equiv 0$ (mod $p$). So $b = 0$ gives the only examples here; this is $U_3$ of Table 2.
Let $V'$ be a minimal proper $Y_1 \times Y_2$-invariant subspace of $V$. Since $X < Y_1 \times Y_2$, $V' = V_1$ or $V_2$; without loss of generality assume $V' = V_2$. The product $Y_1 \times Y_2$ acts irreducibly on $V_2$, so $V_2 = V'_2 \oplus V''_2$ as $Y_1 \times Y_2$-modules, for some $Y_1$-representation $V'_2$ and $Y_2$-representation $V''_2$. By Lemma 2.1, one of $V'_2, V''_2$ must be trivial (since no restricted irreducible $X$-modules are tensor decomposable).

So one of $Y_1, Y_2$ acts irreducibly on $V_2$ while the other acts trivially. Without loss of generality assume $Y_2$ is irreducible on $V_2$ and $Y_1$ acts trivially.

By our assumption that $t$ stabilizes the decomposition $W = W_1 \oplus W_2$, $t$ normalizes $Y_1 \times Y_2$ and interchanges $V_1$ and $V_2$. So, as above, one of $Y_1, Y_2$ acts irreducibly on $V_1$ and the other acts trivially. In particular, if $t$ interchanges $Y_1$ and $Y_2$, then $Y_1$ is irreducible on $V_1$ and $V_2$ on $V_2$; if $t$ preserves each of the factors $Y_1$ and $Y_2$, then $Y_2$ is irreducible on both $V_1$ and $V_2$.

Assume $Y_2$ is irreducible on both $V_1$ and $V_2$. Then $Y$ is irreducible on $V = V_1 \oplus V_2$ and $Y_1 < Y$ acts trivially; thus $Y_1 < Z(Y)$. But $Y$ is simple, so $Z(Y)$ is finite. So $Y_1 = 1 = I(W_1)^t$; so $\dim(W_1) \leq 2$ and $W$ is orthogonal, since $\text{rad}(W_1) = 0$. Then $X \leq Y_2 < Y$ with $(Y_2, Y) = (B_{n-1}, D_n), (D_n, B_n), (D_{n-1}, D_n)$, or $(B_{n-1}, B_n)$, with $V_1$ and $V_2$ both irreducible $Y_2$-modules. In the first and last cases ($Y_2 = B_{n-1}$), we have $X \neq Y_2$ (since $X$ admits a graph automorphism), so either $V_1$ is the natural module for $Y_2$, or the triple $(X, Y_2, V_1)$ appears in Table 1 of [10]. No such triples $(Y_2 = B_{n-1}, X$ admitting a graph automorphism, $V_1 | X$ not symmetric with respect to the graph automorphism) appear in that table, so $V_1$ must be the natural module for $Y_2$, of dimension $2n - 1$. Then $\dim V = 4n - 2$. But $Y = D_n$ or $B_n$, and neither of these groups have any restricted irreducible modules of dimension $4n - 2$ in any characteristic. So these cases don’t occur.

If $(Y_2, Y) = (D_{n-1}, D_n)$, then the same arguments as above apply if $X \neq D_{n-1}$, with the exception that now $\dim V = 4n - 4$, and $D_4$ does have an irreducible restricted module of dimension 16. So $X = A_2$ or $A_3$, and we need to check for irreducible $X$-modules of dimension 8, whose high weights are not symmetric with respect to the graph automorphism. There are no such modules.

So $X = D_{n-1}$. Consider a group of type $B_{n-1}$ sitting between $D_{n-1}$ and $D_n$. If $V |_{B_{n-1}}$ is irreducible, then as at the end of item 2) of case A) above, we are in case $U_3$ of Table 2. If $V |_{B_{n-1}}$ is reducible, let $V'$ be a minimal $B_{n-1}$-invariant subspace. Since $X < B_{n-1}$, $V' = V_1$ or $V_2$, say $V_1$. As above, we check Table 1 of [10] and find that $V_1$ must be the natural module for $B_{n-1}$. But $\dim V_1 = \dim V_2$, so $\dim V = 4n - 2$, which is impossible as above.

Finally, if $X \leq D_n$, we know by Theorem 3.3 what $V | X, V | X^t$ are. By examining Table 1 in [10], we see that there is only one proper connected simple subgroup $X < D_n$ admitting a graph automorphism which is irreducible on any of the $D_n$-modules we obtain in Theorem 3.3; this occurs in $S^t$ of the table. But in this case, $(V | X)^t \cong V | X$, so we have no example So $X = D_n$ is the only case here; this is $U_2$ of Table 2.

So we are left with $Y_1$ acting irreducibly on $V_1$ and $Y_2$ on $V_2$; and since $t$ interchanges them, $Y_1 \cong Y_2$. But now we are exactly in the situation of Lemma 4.3, which tells us that $V \cong W$. □

4.2. Proof of Theorem 4.1 for $G = X \langle t \rangle$. Throughout this section we assume that $G = X \langle t \rangle$, where $t$ is an involutory graph automorphism of $X$.

If $t$ does not act on $W$, then (since $t \in \text{Aut}(Y)$) we must have $Y = A_n$ (if $Y = D_4$, we have chosen $W$ to be the 8-dimensional module fixed by $t$), and we let $A$ be a minimal $X$-invariant subspace of $W$. If $G$ acts irreducibly on $W$, let $A < W$ be a minimal $X$-invariant subspace (so $W = \langle G \cdot A \rangle$). If $G$ acts reducibly on $W$, let $A < W$ be a minimal $G$-invariant subspace. Now either $\text{rad}(A) = A$ or $\text{rad}(A) = 0$.

If $\text{rad}(A) = A$, either $A$ is totally singular, or $p = 2$, $W$ is orthogonal, and $\dim(A) = 1$ (the set of singular vectors in $A$ is an $X$- (or $G$-) invariant subspace, so minimality forces $A$ to be either a 1-space or totally singular). In this exceptional case, we have $W$ even-dimensional and $Y$ of type $D_n$, since if $p = 2$ and $Y = B_n$ we take $W$ to be the symplectic $2n$-dimensional module. So, since $X$ is simple and $I(A)^t = 1$, we have the situation $X \leq B_{n-1} = Y_2 < D_n = Y$, with $Y_2$ being $t$-stable (we must be in the case $A = A'$ here, since otherwise $W = \langle G \cdot A \rangle$ has dimension 2, which would imply $X < A_1$ admits no graph automorphism). As in the proof of Lemma 4.4, there are two possibilities: A. $V$ is irreducible for $B_{n-1}$; or B. $V |_{B_{n-1}} = V_1 \oplus V_2$. If A holds, then as in Lemma 4.4, the only possibility is $U_3$ of Table 2 (by induction from the result in [10]). If B holds, then we again get a contradiction as in the proof of 4.4. So this exceptional case gives only $U_3$. 


Suppose \( A \) is totally singular and \( t \not\in Y \) but \( t \) acts on \( W \), preserving \( A \). Then \( X \leq \text{stab}(A) = P_Y \), a \( t \)-stable parabolic subgroup of \( Y \). So \( P_Y(t) \) acts irreducibly on \( V \). Let \( P_Y = Q_Y L_Y \) be the Levi decomposition; then \( Q_Y \) has fixed points \( V_{Q_Y} \neq 0 \) on \( V \), and \( V_{Q_Y} \) is an irreducible \( L_Y \)-submodule (hence \( P_Y \)-module of \( V \) by the result of [12]). But \( t \) preserves \( Q_Y \), so \( V_{Q_Y} \) is in fact a \( P_Y(t) \)-submodule of \( V \), contrary to our statement above that \( P_Y(t) \) acts irreducibly on \( V \) (since this would imply that \( L_Y \) acts irreducibly on \( V \), which is impossible). If \( A \) is totally singular, \( t \in Y \), and \( A \) is \( t \)-stable, we have \( G \leq \text{stab}(A) \), which is a parabolic subgroup of \( Y \). However, a parabolic subgroup cannot act irreducibly on \( V \).

So if \( A \) is totally singular, we must have either \( W = A \oplus A' \) an irreducible \( G \)-module, or \( t \) not acting on \( W \) (in the second case the form on \( W \) is trivial). If the form on \( W \) is nontrivial, then \( A \) is a maximal totally singular subspace (since \( \dim(A) = \dim(W)/2 \), and \( \text{stab}(A) = P_Y \) is a maximal parabolic subgroup such that (with appropriate choice of Borel subgroup, and switching the labels of \( \alpha_i, \alpha_{i-1} \) if necessary in the \( D_n \) case) \( \Pi(Y) - \Pi(L_Y) = \alpha_j \) (where \( P_Y = Q_Y L_Y \) is the Levi decomposition). In the case \( Y = \text{SL}(W) \), \( \text{stab}(A) \) is a parabolic subgroup \( P_Y \) of \( Y \) such that (with appropriate choice of Borel subgroup) \( \Pi(Y) - \Pi(L_Y) = \alpha_j \) for some \( j \). In both cases, \( X \leq P_Y \). In the case of a nontrivial form let \( j = n \).

Now we want to apply Lemma 4.2; to do so, we need only show that there are only two non-zero \( V'(Q_Y) \), and that they have the same dimension. We do so, that \( X \leq L'_Y \) has only two irreducible summands in \( V \), both of the same dimension. But each quotient \( [V, Q_Y]/[V, Q_Y^{(1)}] \) is \( X < L_Y \)-invariant. So in fact only \( V^1(Q_Y) = V/[V, Q_Y] \) and \( V^2(Q_Y) = [V, Q_Y]/[V, Q_Y, Q_Y] \) can be non-zero; they must have the same dimension because each is isomorphic to one or the other of the \( V_i \), which have the same dimension. So by Lemma 4.2 we only have the possibilities \( V \cong W; Y = D_4 > X = A_3 \) with \( \lambda = \lambda_3 \) with \( V|_X = V|_X(\delta_1) \oplus V|_X(\delta_1) \cong W|_X \); and \( Y = A_n \) for \( n \) odd, \( L'_Y = A_{n-1} \), \( \lambda = \lambda_{n+1}/2 \). Now for the last case we check Table 1 in [10] and see that there are no simple connected proper subgroups of \( A_{n-1} \) (odd) admitting a graph automorphism which act irreducibly on the \( A_{n-1} \)-module with high weight \( \lambda_{n+1}/2 \). So \( X = L'_Y = A_{n-1} \). Similarly no simple connected proper subgroup of \( A_3 \) admitting a graph automorphism acts irreducibly on the \( A_3 \)-module with high weight \( \delta_1 \).

It remains to find \( t: t \not\in Y \) because otherwise \( \dim(A) = \dim(W)/2 = (n + 1)/2 \), \( W = A \oplus A' \), and \( X \) does not stabilize an \( n \)-dimensional subspace (but \( P_Y > X \) does). Let \( t' \) be the standard graph automorphism of \( Y = A_n \). Let \( w_0 \) be the long word in the Weyl group. Then \( t'w_0 \) is an outer automorphism of \( Y \) which stabilizes the Levi factor of \( P_Y \). So \( U_1 \) of Table 2 occurs here.

The automorphism \( t \) for the \( A_3 < D_4 \) case was described in the proof of Lemma 4.2. So \( U_4 \) of Table 2 occurs here.

This completes the proof of the theorem for the case \( \text{rad}(A) = A \).

If \( G = X(t) \) and \( \text{rad}(A) = 0 \), then \( W \) is orthogonal or symplectic and either \( W = A \oplus A' \), \( X \leq I(A)^{\vee} \times I(A')^{\vee} = Y_1 \times Y_2 \), and \( t \) stabilizes \( Y_1 \times Y_2 \), interchanging the two factors; or \( A = A' = A \perp A' = X < I(A)^{\vee} \times I(A')^{\vee} = Y_1 \times Y_2 \), and \( t \) stabilizes each factor. This is the setup of Lemma 4.4, which tells us we must be in one of the situations listed in the Theorem.

We have now proved Theorem 4.1 under the assumption that \( G = X(t) \).

4.3. \( X = D_4 \). Assume \( X = D_4 \) and \( G = X(s) \) or \( X(s,t) \). Note that \( s \in Y \) because no simple group properly containing \( D_4 \) has an outer automorphism of order 3. If \( G = X(s,t) \) and \( V|_X = V_1 \oplus V_2 \), then \( V_1^t = V_1 \), so the \( X \)-high weight of \( V_1 \) is symmetric with respect to \( s \). But then this high weight is symmetric with respect to \( t \) as well, which means that \( V \) is not irreducible as a \( G \)-module (\( \{v \cdot v' \} | v \in V_1 \} \) is a submodule), which is a contradiction. A similar argument shows that whenever \( G \) acts irreducibly on \( W, W|_X \) has either 3 or 6 irreducible summands. Finally, if \( V|_X = V_1 \oplus V_2 \oplus V_3 \) with \( V_3 \) irreducible, then we may assume \( G = X(s) \). So if \( G = X(s,t) \), then \( V \) has six summands as an \( X \)-module.

If \( G \) acts on \( W \), we let \( A \) be a minimal \( G \)-invariant subspace of \( W \) if \( G \) acts reducibly, and a minimal \( X \)-invariant subspace of \( W \) if \( G \) acts irreducibly on \( W \). If \( G = X(s,t) \) and \( t \) does not act on \( W \), we let \( A \) be a minimal \( X \)-invariant subgroup if \( X(s) \) acts irreducibly, and a minimal \( X(s) \)-invariant subspace otherwise.

As before, \( \text{rad}(A) = A \) or \( \text{rad}(A) = 0 \). If \( \text{rad}(A) = A \), either \( A \) is totally singular, or \( p = 2 \), \( W \) is orthogonal, and \( \dim(A) = 1 \) (the set of singular vectors in \( A \) is an \( X \)- (or \( G \))-invariant subspace, so minimality forces \( A \) to be a 1-space or totally singular). As before, in this exceptional case we have \( \dim(W) \) even and \( Y \) of type \( D_n \),
and thus \( t \) acts on \( W \). So, since \( X \) is simple and \( I(A)^1 = 1 \), we have the situation \( X \leq B_{n-1} = Y_2 < D_n = Y \), with \( Y_2 \) being \( t \)-stable (we must be in the case \( A = (G \cdot A) \) here, since otherwise \( W = (G \cdot A) \) has dimension 3 or 6; but neither \( A_5 \) nor \( A_6 \) has a subgroup of type \( D_4 = X \)). We proceed exactly as we did in this case at the beginning of the last section; there are no new examples.

If \( G = D_4(\xi) \) and \( A \) is totally singular, then, as with \( G = X(\xi) \), we have \( W = (G \cdot A) = A \oplus A^2 \oplus A^4 \) irreducible for \( G \) (otherwise \( G \) is contained in a parabolic subgroup of \( Y \)). If the form on \( W \) is nontrivial, then we have an \( X \)-stable chain \( 0 < A < A^2 < A^4 \). Now for any irreducible \( D_4 \)-module we have \( A \cong A^4 \), i.e. \( A^2 \oplus A^4 \cong W/A \) contains a \( D_4 \)-composition factor isomorphic to \( A \). So \( A \cong A^2 \) or \( A^4 \), say \( A^2 \). But then \( A \cong A^2 \cong A^4 \), and \( W = A \oplus A^2 \oplus A^4 \) contains a \( G \)-stable proper subspace \( \{(a,a^\lambda,a^s^\mu) \mid a \in A \} \), contrary to the statement above that \( W \) is irreducible for \( G \). So in fact the form on \( W \) must be trivial; i.e. \( Y = A_n \).

A similar argument for \( G = D_4(\xi,t) \) shows that \( W \) must be irreducible for \( G \) and the form on \( W \) must be trivial when \( t \) acts on \( W \). If \( s = 2 \) does not act on \( W \), we already know the form must be trivial, but we must consider the case when \( W \) is reducible for \( X(\xi) \). Assume this is the case; then \( A \) is a minimal \( X(\xi) \)-invariant subspace of \( W \), so \( X(\xi) \leq \text{stabilizer}(A) = P_t \) a parabolic subgroup of \( Y \). But then \( P_t \) cannot act irreducibly on \( V \) and \( X(\xi) \) has only two irreducible summands in \( V \). Lemma 4.2 applies. We see that we must have \( Y = A_n, L^2 = A_{n-1}^\perp, \lambda = h_{(n+1)/2} \) and the restriction of \( V \) to \( L^2_n \). Then \( G = X(\xi), Y_1 = L^2_1, \) and \( V_1 = V_1(h_{(n+1)/2}) \). Then we inductively have the situation we are examining in this section: \( G_1 < Y_1, V_1 \) is an irreducible restricted \( Y_1 \)-module which is also irreducible for \( G_1 \) but not for \( X \), with restricted \( X \)-high weights. Since there are no examples for this setup, we have none for \( G \leq Y \).

So \( Y = A_n \) and \( W \) is irreducible for \( G \) (or for \( X(\xi) \) when \( t \in G \) does not act on \( W \)). Consider the action of \( X = D_4 \) on \( W \). Via the isomorphism \( A \cong X \cong A^4 \). \( X \) fixes a nondegenerate form on \( A \) (this form is orthogonal if \( p \neq 2 \) by [14, Lemma 79]). Similarly, \( X \) fixes a form of the same type on each of its other summands in \( W \), given by the action of \( s \) (and \( t \) if necessary): \((a^s,b^s) = (a,b)\) for \( a,b \in A \). Define a form on \( W \) by setting \((a,b) = 0\) for \( a,b \) in different summands; then \( X \) fixes this form. If the form is orthogonal, choose an orthonormal basis for \( A \); translate this basis by \( s \) (and \( t \)) to obtain bases of \( A^2 \) and \( A^4 \) (and \( A^2, A^4, A^4 \) if \( t \in G \) and \( t \) acts on \( W \)). The union of these bases is an orthogonal basis for \( W \), and by Witt’s Theorem there is an element \( \xi \in SO(W) \) which permutes the \( X \)-summands of \( W \) as \( s \) does. So in fact \( X(\xi) \cong X(\xi) \) fixes some form, contrary to \( Y = A_n \) (we always take \( Y \) to be the smallest of \( SL(W), SO(W), Sp(W) \)) which contains \( X \) and whose automorphism group contains \( G \). If the form on \( A \) is symplectic (as noted above, this can only happen for \( p = 2 \)), then choose a hyperbolic basis on each \( X \)-summand of \( W \) and continue as above.

This completes the argument for \( rad(A) = (A) \).

If \( \text{rad}(A) = 0, X = D_4 \) and \( G = X(\xi) \) or \( G = X(\xi,t) \), then one of the following holds (notice that \( t \) must act on \( W \) if it is in \( G \), since \( Y \neq A_n \)): 

**A** \( A = (G \cdot A) \) and \( W = A \perp A^2 \), \( X \leq I(A) \times I(A^1) = Y_1 \times Y_2 \). As \( s \in Y \) and \( s \) preserves \( A \) and \( A^2 \), in fact \( X(\xi) \leq I(A) \times I(A^1) \). So either \( V|_{Y_1 \times Y_2} \) is irreducible, or \( V|_{Y_1 \times Y_2} \) has two summands, interchanged by \( t \) (\( t \) preserves \( A \) and so acts on \( Y_1 \) and \( Y_2 \)).

If \( V|_{Y_1 \times Y_2} \) is irreducible, then the possibilities for \( V|_{Y_1 \times Y_2} \) are as on page 16. In case 1) there, \( V = V' \otimes V'' = V_1 \otimes \cdots \otimes V_6 \) (\( k = 3 \) or 6) as \( X \)-modules, for some \( X \)-modules \( V' \) and \( V'' \), which must both be reducible as \( X \)-modules as on page 16.

If \( V \) is a \( T_X \)-high weight of \( V' \) and \( \mu_1 \) a \( T_X \)-high weight of \( V'' \), then \( \mu_1 + \mu_2 \) is a \( T_X \)-high weight of \( V' \otimes V'' \). In our case, since \( s \) preserves \( A \), it acts on \( V' \) and \( V'' \). So \( \mu_1^s \) and \( \mu_2^s \) are also high weights of \( V' \). If \( \mu_1 \) and \( \mu_2 \) are not symmetric with respect to \( s \), then it is easy to see that the possible sums \( \mu_1^s + \mu_2^j \) for \( 0 \leq i, j \leq 2 \) are not all images under \( (s,t) \) of a single \( T_X \)-weight. If both \( \mu_1 \) and \( \mu_2 \) are symmetric with respect to \( s \), then so is \( \mu_1 + \mu_2 \) — but \( V \) has no \( T_X \)-high weights which are symmetric with respect to \( s \). Finally, if \( \mu_1 \) is not symmetric with respect to \( s \), and all the \( T_X \)-high weights (there are at least two, say \( \mu_2 \) and \( \mu_2^s \)) of \( V'' \) are symmetric, we see that either \( \mu_1 + \mu_2 \) and \( \mu_1 + \mu_2^s \) are not conjugate under the action of \( (s,t) \) (if \( \mu_2 \neq \mu_2^s \)), or there is a \( T_X \)-high weight space in \( V' \) of dimension bigger than 1. Neither is possible.

If \( V|_{Y_1 \times Y_2} \) are as in case 2) on page 16, then we have \( D_4(\xi) < B_n < D_{n+1} \). In fact, though, since \( t \) also acts on \( B_n \) (if \( t \in G \) and \( B_n \) has no outer automorphisms, we have \( G < B_n \)). In [5] it is shown that there are no examples with \( X \) acting irreducibly on the natural module for \( B_n \). So \( X \) must act reducibly on the natural.
module for $B_n$; i.e. we are back in the situation we consider here. So we have no examples, as $B_n$ doesn’t appear as an overgroup for $D_4(s)$.

If $V|_{Y_1 \times Y_2}$ is reducible, then $t \in G$ and $V|_{Y_1 \times Y_2}$ has two summands, interchanged by $t$ ($t$ normalizes $Y_1 \times Y_2$). Let $V'$ be one of these summands; then, renumbering if necessary, $V' = V_1 \oplus V_2 \oplus V_3$. So $V_1 \oplus V_2 \oplus V_3 = V' \otimes V^2$ for some $Y_1$-module $V^1$ and $Y_2$-module $V^2$.

These modules $V^1$ and $V^2$ must be reducible as $X$-modules, since otherwise the fact that $s$ acts on them would force the $T_X$-high weight $\mu_i$ of $V'$ to be $s$-symmetric, in which case the $T_X$-high weight $\mu_1 + \mu_2$ of $V$ is $s$-symmetric, which is a contradiction.

Now we argue as above and conclude that one of the $V^i$, say $V^2$, must be trivial. But then $(G_0 = X(s), Y_1, V^1)$ must be an example appearing in Table 2; since there are none of this form, we have no examples here.

B) $W$ is irreducible as a $G$-module and $A$ is an irreducible $X$-module. Then $W = A \oplus A' \oplus A''$ ($\oplus A' \oplus A'' \oplus A''$, possibly) = $W_1 \oplus \cdots \oplus W_l$ and $X \leq I(A) \times I(A') \times I(A'')$ ($\times I(A'') \times I(A'') \times \cdots \times I(A'')) = Y_1 \times \cdots \times Y_l$. Recall $V = \chi V_1 \otimes \cdots \otimes V_k$ ($k = 3$ for $G = X(s)$, 6 for $G = X(x,t)$).

We can see by examining Table 1 of [10] that there are no possibilities here for $V|_{Y_1 \times \cdots \times Y_l}$ irreducible.

So $V$ is reducible $Y_1 \times \cdots \times Y_l$-module; let $V'$ be a $Y_1 \times \cdots \times Y_l$-stable subspace of minimal dimension. Since $X < Y_1 \times \cdots \times Y_l$, this implies that $V' = \sum_{i \in I} V_i \subseteq \{1, \ldots, k\}, |I| \leq k/2$ (since $t$ and $s$ normalize $Y_1 \times \cdots \times Y_l$, $V''$ is another $Y_1 \times \cdots \times Y_l$-stable subspace). If $V' = V_1$ (i.e. $|I| = 1$), then we have $V_1 = \chi V_1^{(1)} \otimes \cdots \otimes V_1^{(l)}$ for some $Y_i$-modules $V_1^{(i)}$. But restricted irreducible $D_4$-modules are not tensor decomposable (2.1), so only one $V_1^{(i)}$ is nontrivial, say $V_1^{(1)}$; i.e. only $Y_1$ acts nontrivially on $V_1$. But $G$ permutes the $V_i$ and the $Y_i$ and we see that for every $j$, $V_j$ is acted on irreducibly by one $Y_i$ trivially by the others. Then we are in the situation of Lemma 4.3, which tells us $V \cong W$.

If $|I| > 1$, we must be in the case $k = 6$ (since $|I| \leq k/2$). We have (perhaps renumbering the $V_i$) $V_1 \oplus V_2 \oplus V_2(\oplus V_3) = V^{(1)} \otimes \cdots \otimes V^{(l)}$ for $Y_i$-modules $V^{(i)}$. If one of the $V^{(i)}$ is reducible as an $X$-module (via the projection $X \to Y_i$), we conclude as above that only one of the $V^{(i)}$ is nontrivial and apply Lemma 4.3, as otherwise we could conclude that one of the $V_j$ was tensor decomposable as an $X$-module.

So we assume that each $V^{(i)}$ is irreducible as an $X$-module, and at least two of them are nontrivial. As $s$ and $t$ permute them, all the $Y_i$ are isomorphic. Note that $s$ and $t$ act on $Y_1 \times \cdots \times Y_l$, and again since $X \leq Y_1 \times \cdots \times Y_l$, $V$ is irreducible for $(Y_1 \times \cdots \times Y_l)(s,t)$. So if $V' = V_1 \oplus V_2$, this implies that $V = V' \oplus V'_{\sigma} \oplus V'_{t\sigma}$, with $t$ stabilizing $V'$ and interchanging $V_1$ and $V_2$. In this case $t$ must permute the $Y_i$ which act nontrivially. If $V' = V_1 \oplus V_2 \oplus V_3$, we have $V = V' \oplus V_3^{(s)}$, with $s$ stabilizing $V'$ and permuting $V_1$, $V_2$, and $V_3$, and permuting the $Y_i$ which act nontrivially on $V'$.

Note that if $l = 6$, since $G$ acts irreducibly on $W$, all six $X$-summands of $W$ must be non-isomorphic. In other words, the labelling of the $T_X$-high weight $\gamma$ of $W_1 = A$ (which is the natural module of $Y_1$) must be non-symmetric with respect to the action of $s$ and $t$ on the Dynkin diagram for $X$ (if $\gamma = \gamma'$ for $\sigma$ the induced action of $s$ or $t$ on the Dynkin diagram, then $G$ preserves a “diagonal” submodule of $W$). Similarly, if $l = 3$ then $\gamma$ cannot be of the form $a\delta_1 + b\delta_2 + a\delta_3 + a\delta_4$.

Now consider the possibilities for the $V^{(i)}$. The group $D_4$ appears only once in Table 1 of [10] as a subgroup of a classical group other than $A_n$ (case $S_8$ there), and in that case the restriction of the natural module of $Y_i$ to $D_4$ has a symmetric high weight. Thus either each of the nontrivial $V^{(i)}$ is the natural module for $Y_i$, or $X = D_4 \cong Y_i$ for every $i$.

Assume that each of the nontrivial $V^{(i)}$ is the natural module for $Y_i$. If $V' = V_1 \oplus V_2$, we know that $t$ preserves $V'$, permuting the $Y_i$ which act nontrivially. But the high weights of the natural modules of $Y_i$ and $Y_i'$ are $t$-conjugate; in other words, their sum is symmetric with respect to $t$. Since the sum of the high weights for the $V^{(i)}$ is a high weight in $\otimes V^{(i)}$, this gives a $t$-symmetric high weight in $V'$, which is impossible as $t$ interchanges the two $T_X$-high weights there. Similarly if $V' = V_1 \oplus V_2 \oplus V_3$, then $s$ acts on $V'$ and the high weights of the natural modules of $Y_1$, $Y_1'$, and $Y_1''$ add up to an $s$-symmetric high weight in $V'$, which is again impossible.
So the only setup we have not ruled out is $X = D_4 \cong Y_1$ for every $i$. In this case, the natural module for $Y_1$ restricts to $X$ as a natural module for $D_4$, whose high weight does not have six $S_3$-conjugates. So we know that $l = 3$ with $r$ fixing one of $Y_1$, $Y_2$, and $Y_3$ and interchanging the other two, and $Y = D_{12}$. If $V' = V_1 \oplus V_2$ then $t$ stabilizes $V'$ and so the high weights of the $V^{(i)}$ add up to a $t$-stable high weight in $V'$, which is impossible as above.

If $V' = V_1 \oplus V_2 \oplus V_3$, then $s$ stabilizes $V'$, we have $V = V' \oplus V''$, and $s$ permutes the $Y_i$ which act nontrivially on $V'$ (which implies all three $Y_i$ act nontrivially). But then the $T_X$-high weights of the $V^{(i)}$ are $s$-conjugates of each other, and their sum is symmetric with respect to $s$, which again is a contradiction.

This completes the proof of Theorem 4.1. \hfill \Box

5. THE CASE WHEN $t$ DOES NOT ACT ON $W$

In this section we consider the cases when $t$ acts irreducibly on $W$, $t \in G$, and $t$ does not act on $W$.

Notice that if $G = D_4 \langle s,t \rangle$ and $D_4$ acts irreducibly on $W$, then the fact that $s \in Y$ forces the $D_4$-high weight of $W$ to be of the form $a_0t + b_0s + a_2s + a_4$, which implies that $t$ also acts on $W$. So we need consider only $G = X \langle t \rangle$. The main result is:

**Theorem 5.1.** If $G = X \langle t \rangle$, $X$ acts irreducibly on $W$, and $t$ does not act on $W$, then we are in situation $U_7$, $U_8$, or $U_9$ of Table 2, all of which occur.

**Proof.** We will use heavily the construction given in Lemma 2.7 of a parabolic subgroup of $Y$ containing a given parabolic subgroup of $X$ (we usually apply it to the Borel subgroup $B_X$). First we need a Lemma about that embedding; this will be useful in [5] as well:

**Lemma 5.2.** If $P_Y$ is a $t$-stable parabolic subgroup of $Y$ such that $B_Y < P_Y$, $U_X < Q_Y$, $T_X < L_Y$ (where $P_Y = Q_YL_Y$, $B_Y = U_XT_X$ are Levi decompositions), then one of the simple factors of $L_Y$ has type $A_1$; and if this factor corresponds to $\alpha_i$, then $a_i = 1$. In addition, $a_i = 0$ for $\alpha_i \in \Pi(L_Y^s)$, $i \neq j$.

**Proof.** With such a setup, 2.10 in [10] implies that $V/[V,Q_Y]$ is an irreducible $L_Y$-module.

Since $P_Y$ is $t$-stable, $[V,Q_Y]$ is $T_X \langle t \rangle$-stable, hence $[V,Q_Y] \cong [V,U_X]$ since $U_X \leq Q_Y$, and we have equality because $V/[V,U_X]$ is an irreducible $T_X \langle t \rangle$-module and $T_X \leq L_Y$). So

$$V/[V,Q_Y] = V/[V,U_X] = V_i/[V_1,U_X] \oplus V_2/[V_2,U_X],$$

which has dimension 2 by Lemma 2.4. So some simple factor of $L_Y^s$ has type $A_1$, the marking on the corresponding node of the Dynkin diagram in the labelling for $\lambda$ is 1, and all other markings are 0. \hfill \Box

Now embedding a Borel subgroup $B_X$ of $X$ in a parabolic subgroup of $Y$ via the $U_X$-level construction of Lemma 2.7 gives a parabolic subgroup $P_Y$ which by Lemma 2.8 is $t$-stable. Let $P_Y = Q_YL_Y$ be the Levi decomposition produced in the construction of Lemma 2.7.

The group $Y$ is a simple algebraic group of classical type, and either $t \in Y$ or $t$ is an outer automorphism of $Y$. If $Y$ is of type $B_n$ or $C_n$, then $Y$ has no outer automorphisms, so $t \in Y$ and $t$ acts on $W$. If $Y = D_n$ and $t$ is outer, then $t$ acts on the natural module $W$ (which has high weight $\lambda_1$). Since we are assuming $t$ does not act on $W$, we must therefore have $Y = A_n$.

Since $t$ acts on $W$, we have a $t$-symmetric $T_Y$-high weight for $V$, which imposes strong restrictions as $Y = A_n$.

By Lemma 5.2, one of the factors $L_i$ or $L_i'$ must be of type $A_1$, and if this factor corresponds to $\alpha_i \in \Pi(Y)$, then $a_i = 1$ for every $i$ such that $\alpha_i \in \Pi(L_i^s)$. The fact that $P_Y$ is $t$-stable implies that if $\alpha_i$ corresponds to an $A_1$-factor of $L_i$, then so does $\alpha_i \in \Pi(L_i^s)$. Since $\lambda$ is also $t$-stable, $a_i = a_{n-j+1}$. But this says that either there are two non-zero labels on $\Pi(L_i^s)$ (which is impossible by Lemma 5.2), or $j = n - j + 1$. So we must in fact have an $L_i$-factor of $L_i^s$ corresponding to $\alpha_{n-j+1}$ (in particular, $n$ must be odd).

We noted in the proof of Lemma 2.8 that the dimension of $U_X$-level $l$ of $W$ is the same as the dimension of $U_X$-level $l^k - i$, where $l^k$ is the level of the low weight ($l^k$ is also minimal with respect to $W,U_X$). This implies that the $U_X$-level corresponding to the $A_1$-factor with $\Pi(L_i^s) = \{\alpha_{n-j+1}\}/2$ must be level $l^k/2$. So to prove that a given high weight $\delta$ for $W|X$ gives no examples, it suffices to show that $W_{l^k/2}$ has dimension different than 2. In particular, if $l^k$ is odd, we have nothing to check, since then level $l^k/2$ does not exist.
We rely heavily on the results in [15] (for $X = A_m$) and [9] (for $X = D_m$ and $X = E_6$) that all weights which appear in the Weyl module for $X$ with high weight $\delta$ also appear in $W = V_X(\delta)$. We use induction on the height of $\delta$ in the weight lattice, based on the fact that if $\mu$ is a weight in $V_X(\delta)$ (i.e. $V_X(\delta)_{\mu} \neq 0$) at level $l$, and $\nu$ is a weight in $V_X(\mu)$ at level $k$, then $\nu$ is a weight in $V_X(\delta)$ at level $l + k$. Using this fact, the inductive step is trivial.

5.1. $X = A_m$. Assume $X$ is of type $A_m$. Let $\delta = d_1\delta_1 + d_2\delta_2 + \cdots + d_m\delta_m$ ($d_i \geq 0$) be the $T_X$-high weight of $W$. Since $t$ does not act on $W$, $d_i = d_{m-i+1}$ for some $i$. First we will establish that $\delta$ has a fundamental dominant weight or 0 as a subdominant weight; following that we will show that $A_{m'}$ modules with fundamental dominant weights have at least 3 weights at the “middle” level except in a few cases; finally, we will deal with these few cases and with the case $\delta \not\succ 0$.

**Lemma 5.3.** Either $\delta \succ \delta_i$ for some $i$, $1 \leq i \leq m$, or $\delta \succ 0$.

**Proof.** See exercise III.13 in [7].

So if we show that $V_X(\delta_i)$ has three or more weights at level $l_i/2$, then all $\delta$ with $\delta_i$ as a subdominant weight will be ruled out. The next step is:

**Lemma 5.4.** If $\delta$ is a non-zero dominant weight for $X$ which is not symmetric with respect to the action of $t$, such that level $l_\delta/2$ has exactly two non-zero weights, then we are in one of the following situations (perhaps after reversing the labelling):

1. $X = A_2$, with $\delta = 3\delta_1$, $\delta = 2\delta_2$, $\delta = 2\delta_1 + \delta_2$, or $\delta = 3\delta_1 + \delta_2$;  
2. $X = A_3$, with $\delta = 2\delta_1$; or  
3. $X = A_4$, with $\delta = \delta_2$.

**Proof.** The proof is long and somewhat tedious; we give a sketch here. By the last lemma, $\delta$ has some $\delta_i$ or 0 as a subdominant weight. We may assume $i \leq (m + 1)/2$ by reversing the labelling if necessary.

If we restrict the usual ordering on the weights of $T_X$ ($\mu \succ \nu$ if $\mu - \nu$ is a sum of positive roots) to the dominant weights, then some of the weights in question have the nice property that they have unique immediate successors; e.g. $\delta_1 \prec \delta_i + \delta_{m+1}$; $0 \prec \delta_1 + \delta_{m} \prec \delta_2 + \delta_{m-1}$ ($m \geq 3$), or $\delta_1 \prec \delta_2 \prec \delta_1 + \delta_3$ ($m = 2$). We use this fact, the last lemma, and the fact that $\delta$ is not $t$-symmetric, to exhibit at least 3 weights (or show that there is at most 1 weight) at the middle level when $\delta$ is not one of the weights listed in the lemma.

So we need consider only those $\delta$ listed in the statement of the Lemma, as the middle level must be of dimension 2, and the only $\delta$ with a single weight at the middle level have a weight space of dimension 1 for that weight.

(1) Assume $X = A_2$ and $\delta = 3\delta_1$. Then $Y$ is of type $A_2$, and if $P_Y = Q_Y L_Y$ is the parabolic subgroup containing the Borel subgroup $B_X$ of $X$, given as the stabilizer of the $U_X$-levels of $W$, then $P_Y$ corresponds to $\{\alpha_3, \alpha_5, \alpha_2\} \subseteq \Pi(Y)$. For $\lambda = \sum a_i\alpha_i$, the $T_Y$-high weight of $V$, by Lemma 5.2 and the comments which follow it, we have $a_5 = 1$ and $a_3 = a_2 = 0$. Now by Lemma 2.9, we have $[V, Q_Y] = [V, Q_X]$ and, since $Q \leq Q_Y$, we have $[V, Q_X, Q_Y] \leq [V, Q_Y, Q_Y]$. Thus $\dim([V, Q_Y]/[V, Q_Y, Q_Y]) = \dim(V^2(Q_Y)) = \dim(V^2(Q_X))$. Then applying Lemma 2.4 to $Q_X$, we have $\dim(V^2(Q_Y)) \leq \dim(V^2(Q_X)) \leq 2 \dim(V^2(Q_X)) = 2 \dim(V^2(Q_Y)) = 4$.

But then $a_1 > 0$ for some $i \neq 5$ violates this bound. So in fact $V = \bigwedge^3(W)$, of dimension $\binom{10}{3}/2 = 252$.

A high weight vector of $\bigwedge^3(W|X)$ is $w_1 \wedge w_2 \wedge w_3 \wedge w_4 \wedge w_5$, where $w_1 \in W_6$, $w_2 \in W_6\beta_1$, $w_3 \in W_6\beta_2$, $w_4 \in W_6\beta_1\beta_2$, and $w_5 \in W_6\beta_1\beta_2\beta_3$. So a $T_X$-high weight of $V$ is $5\delta - 6\beta_1 - 2\beta_2 = 5\delta_1 + 2\delta_2$. The dimension of the Weyl module with this high weight is 81, so $\dim(V_1) \leq 81 < 126 = \frac{3}{2}\dim(V)$, which is a contradiction. So $\delta = 3\delta_1$ gives no examples.

If $\delta = 2\delta_2$, then $Y$ is of type $A_3$ and $P_Y = Q_Y L_Y$ as above corresponds to $\alpha_3 \in \Pi(Y)$. As above, any non-zero $a_i$ other than $a_3 = 1$ contradicts $\dim(V^2(Q_Y)) \leq 4$ (since $\lambda$ is symmetric with respect to the action of $t$). So $V = \bigwedge^3(W)$, of dimension 20. As above, we can compute the weight of a $T_X$-high weight vector of $V$: we find that it is $3\delta_1$ or $3\delta_2$. The $A_2$-modules with these high weights each have dimension $10 = \frac{1}{2}\dim(V)$ if $p \neq 2, 3$. So here we have case U7 of Table 2: $p \neq 2, 3$, $X = A_2$, $Y = A_5$, $V_1 = V_{A_2}(3\delta_1)$, and $V = V_{A_5}(\lambda_3)$. 


If \( \delta = 2\delta_1 + \delta_2 \), then although there are only 2 weights at the middle level (\( \delta - \beta_1 - 2\beta_2 \) and \( \delta - 2\beta_1 - \beta_2 \)), one of them (\( \delta - 2\beta_1 - \beta_2 \)) has a weight space of dimension 2 in all characteristics. So \( \dim(W) = 3 \), and there is no \( A_1 \)-factor of \( L_\gamma \) corresponding to the “middle” level.

Finally, if \( \delta = 3\delta_1 + \delta_2 \), then as above the weights in the middle level have weight spaces of dimension 2 or more except in characteristic 5. So for \( p \neq 5 \) there are no examples here. If \( p = 5 \), then \( Y \) has type \( A_17 \), and \( a_0 = 1 \) is the only non-zero \( a_i \) for \( \alpha_i \in \Pi(L_\gamma) \). Now if some \( a_i \neq 0 \) for \( \alpha_i \in \Pi(Y) - \Pi(L_\gamma) \), we obtain a contradiction to \( \dim(V^2(Y)) \leq 4 \). So in fact \( V = \bigwedge^9(W) \), of dimension 48,620. But no restricted \( A_2 \)-module has dimension 24,310 (Weyl’s character formula shows that the dimension of the Weyl module with a restricted high weight is at most the dimension of the Weyl module with high weight \( (p - 1)p \), and this dimension is 125). So there are no examples here.

(2) Suppose \( X = A_3 \) and \( \delta = 2\delta_1 \). Then \( Y \) is of type \( A_9 \) and \( a_5 = 1 \). By Lemmas 2.4 and 2.9 as before, \( \dim(V^2(Y)) \leq \dim(V^3(Y)) \leq 3 \dim(V^1(QY)) = 3 \dim(V^1(Y)) = 6 \), when \( p_Y = Q_Y \cdot L_Y \) is the parabolic subgroup of \( Y \) containing \( B_Y \), constructed as the stabilizer of the \( U_X \)-levels of \( W \). In this situation \( p_Y \) corresponds to \( \{\alpha_0, \alpha_5, \alpha_9\} \subseteq \Pi(Y) \). Then \( \dim(V^2(Y)) \leq 6 \) implies \( a_i = 0 \) for \( i \neq 5 \), as otherwise too many non-zero weight spaces appear in \( V^2(Y) \). So \( V = \bigwedge^5(W) \), of dimension 252; as above, we can compute a \( T_X \)-high weight of \( V \) and obtain a contradiction to \( \dim(V^2(Y)) \leq 4 \). So again, \( V = \bigwedge^5(W) \), of dimension 252. Again we compute a high weight of \( \bigwedge^5(W|_X) \), obtaining \( \delta_2 + 2\delta_4 \) or \( 2\delta_2 + \delta_3 \). The \( A_3 \)-modules with these high weights have dimension 126 except in characteristics 2 and 5. So here we have the last example given in Table 2.

This completes the examination of the cases for \( X = A_m \).

5.2. \( X = D_m \). We must establish a result analogous to the first lemma of the last subsection, narrowing the range of possibilities for \( \delta \) we must examine.

**Lemma 5.5.** If \( \delta \neq \delta_1 \) is a non-zero dominant weight of \( T_X \), then \( \delta \) has one of \( \delta_2, \delta_3, \delta_{m-1}, \) or \( \delta_m \) (or \( \delta + \delta_4 \) if \( X = D_4 \)) as a subdominant weight.

**Proof.** By exercise III.13 in [7], \( \delta \) has one of \( 0, \delta_1, \delta_{m-1}, \) or \( \delta_m \) as a subdominant weight.

The unique successor to \( 0 \) in the partial order on the dominant weights is \( \delta_2 \), so if \( \delta \succ 0 \), then \( \delta \succ \delta_2 \) (since \( \delta \neq 0 \)).

Assume \( \delta \succ \delta_1 \). We assumed \( \delta \neq \delta_1 \), and adding positive roots to get a dominant weight forces \( \delta \succ \delta_3 \succ \delta_1 \) if \( m > 4 \), or \( \delta \succ \delta_3 + \delta_4 \) if \( m = 4 \). So in fact \( \delta \succ \delta_3 \).

Note that \( \delta = \delta_1 \) and \( \delta = \delta_2 \) are not possibilities, since \( \delta \) is not symmetric with respect to \( t \) (as \( t \) does not act on \( W \)). The two immediate successors to \( \delta_2 \) in the partial order on the dominant weights are \( 2\delta_1 + \delta_4 \) and \( \delta_4 + \delta_5 \) (or, if \( m = 4 \), \( 2\delta_3 \) or \( 2\delta_4 \); if \( m = 5 \), \( \delta_4 + \delta_5 \)). So if \( m > 5 \) and \( \delta \succ \delta_2 \), then \( \delta \succ 2\delta_1 \) or \( \delta \succ \delta_4 \). If we can show that \( V_{D_3}(\delta_1 + \delta_4), V_{D_3}(2\delta_4), V_{D_4}(\delta_4 + \delta_5), \) and \( V_{D_m}(\delta_1) \) for \( i \in \{3, 4, m - 1, m\} \) have at least three weights at their middle levels, we will be done.

The first possibilities listed (\( 2\delta_1 \) and \( \delta_1 + \delta_4 \)) for \( m = 4 \), \( \delta_4 + \delta_5 \) for \( m = 5 \), \( 2\delta_1 \), \( \delta_3 \), and \( \delta_4 \) are relatively easily dealt with by simply listing three weights at the middle level. For example, in \( V_X(\delta_3) \) the low weight is at level \( 6m - 12 \), so the middle level is \( 3m - 6 \). At this level, for \( m \geq 6 \) we have the weights \( \pm(\delta_{m-1} - \delta_m) \) (by first subtracting \( \beta_3 + \cdots + \beta_{m-1} \), then \( \beta_2 + \cdots + \beta_{m-2} + \beta_m \), and finally \( \beta_1 + \cdots + \beta_{m-1} \) or \( \beta_1 + \cdots + \beta_{m-2} + \beta_m \)), and

\[
\delta_3 - 2\beta_1 - 2\beta_2 - 3\beta_3 - 3\beta_4 - \cdots - 3\beta_{m-2} - \beta_{m-1} - \beta_m = -2\delta_1 + \delta_2 - \delta_{m-2} + \delta_{m-1} + \delta_m.
\]

The cases for \( m \leq 5 \) and the other weights are similar.

Listing three weights at the middle level for \( \delta = \delta_m \) is more difficult (\( \delta = \delta_{m-1} \) is the same argument by symmetry). The low weight here is at level \( m(m - 1)/2 \), so we must check level \( m(m - 1)/4 \). The level only
exists if \( m = 4k \) or \( m = 4k + 1 \) for some \( k \). For \( m = 8 \) or 9, we can check directly, writing down at least 3 weights at this middle level. So assume \( m \geq 12 \). Then

\[
\delta' = \delta_m - (\beta_2 + \cdots + \beta_{m-2} + \beta_m) - (\beta_3 + \cdots + \beta_{m-1})
\]

\[
= \delta_1 - \delta_3 + \delta_m
\]

is a weight of \( V_X(\delta_m) \) at level \( 2m - 5 \). Let \( P_X \leq X \) be the obvious parabolic subgroup with Levi factor \( L \) such that \( L' \) is of type \( D_{m-4} \), and let \( \{ \gamma_1, \ldots, \gamma_{m-4} \} \) be the fundamental dominant weights of \( L' \). Then \( \delta'|_{T_L} = \delta_m|_{T_L} = \gamma_{m-4} \), and by induction this \( D_{m-4} \)-weight has at least three weights at level \((m-5)(m-4)/4 \). This implies \( \delta' \) has at least three weights at this level, which in turn implies that \( \delta_m \) has at least three at level \((2m - 5) + (m - 5)(m - 4)/4 = m(m - 1)/4 \), which is its middle level. So if \( \delta \gg \delta_m \) for \( m \geq 6 \), then \( \delta \) has at least 3 weights at its middle level.

For \( m = 4 \) or 5, \( V_X(\delta_m) \) has only two weights, both with weight spaces of dimension 1, at level \( m(m - 1)/4 \).

Assume \( m = 4 \). Then \( Y \) has type \( A_7 \), and when we embed the Borel subgroup \( B_X \) in a parabolic subgroup \( P_Y = Q_Y L_Y \) of \( Y \), the subgroup we obtain corresponds to \( \alpha_4 \in \Pi(Y) \). So \( \alpha_4 = 1 \). By Lemmas 2.4 and 2.9, we have \( \dim(V^2(Q_Y)) \leq \dim(V^2(Q_X)) \leq 4 \cdot \dim(V^1(Q_X)) = 4 \cdot \dim(V^1(Q_Y)) = 8 \). Since \( \delta \) acts on \( V \), we have \( a_1 = a_2 = a_4, a_2 = a_6, \) and \( a_3 = a_5 \). Now if \( m \geq 6 \), then \( \delta \gg \delta_m \) for \( m \geq 6 \), then \( \delta \) has at least 3 weights at its middle level.

Next consider the embedding of the \((t, \text{stable})\) parabolic subgroup \( P_X = Q_X L_X \) of \( X \) corresponding to \( \{ \beta_2 \} \subseteq \Pi(X) \) in a parabolic subgroup \( P_Y = Q_Y L_Y \) of \( Y \) via \( Q_X \)-levels. Here \( P_Y \) corresponds to \( \{ \alpha_3, \alpha_5 \} \subseteq \Pi(Y) \). By Lemma 2.7, we have \( V/[V, Q_Y] = V/[V_1, Q_X] \oplus V_2/[V_2, Q_X] \), we can compute dimensions (since an irreducible restricted \( A_1 \)-module has the same dimension as the Weyl module with the same high weight) and obtain \( 2(b_1 + 1) = (a_1 + 1)(a_3 + 1) = (a_1 + 1)^2 \), or \( b_1 = \frac{(a_1 + 1)^2}{2} - 1 \). But \( b_1 \) is integral, so this implies \( a_3 \neq 0 \) (indeed, \( a_3 \) must be odd). So \( a_1 = a_2 = 0 \) by our above comment that only one of these three can be non-zero.

Our penultimate \( t \)-stable parabolic subgroup to consider in this case is the one corresponding to \( \{ \beta_2 \} \subseteq \Pi(X) \). When we embed this via \( Q_X \)-levels, we have \( P_Y = Q_Y L_Y \) corresponding to \( \{ \alpha_2, \alpha_4, \alpha_6 \} \subseteq \Pi(Y) \). Here again, by Lemmas 2.4 and 2.9, we have \( \dim(V^2(Q_Y)) \leq \dim(V^2(Q_X)) \leq 6 \cdot \dim(V^1(Q_X)) = 6 \cdot \dim(V^1(Q_Y)) = 12 \) (we have \( \dim(V^1(Q_Y)) = 2 \) because we discovered above that \( a_2 = a_6 = 0 \)). Since \( a_3 \neq 0 \), we have the high weights \( \lambda - \alpha_3 \) and \( \lambda - \alpha_4 \) in \( V^2(Q_Y) \), each giving \( L'_Y \)-modules of dimension 6 (unless \( p = 2 \), in which case each has dimension 4). If \( a_3 \neq 2 \), then \( \lambda - \alpha_3 - \alpha_4 \) has a 2-dimensional weight space in \( V \), which implies that \( \lambda - \alpha_3 - \alpha_4 \) is another high weight in \( V^2(Q_Y) \). But this contradicts \( \dim(V^2(Q_Y)) \leq 12 \), if \( p \neq 2 \).

So \( a_3 = p - 2 \), or \( p = 2 \). Assume \( a_3 = p - 2 \). Above we had \( b_1 = \frac{(a_1 + 1)^2}{2} - 1 \). Since \( b_1 \leq p - 1 \), this gives \( \frac{(p - 1)^2}{2} - 1 \leq p - 1 \), or \( p \leq 4 \). So if \( 0 \neq a_3 = p - 2 \), we have \( p = 3 \) and \( a_3 = 1 \); otherwise \( p = 2 \) and \( a_3 = 1 \). The \( p = 2 \) case cannot occur as the \( A_7 \)-module with this high weight has dimension 64512; the largest restricted irreducible \( D_4 \)-module has dimension 4096.

Now for one last \( t \)-stable parabolic subgroup of \( X \): Let \( P_X \) correspond to the subset \( \{ \beta_2, \beta_3, \beta_4 \} \) of \( \Pi(X) \). When we embed this in a parabolic subgroup \( P_Y \) via \( Q_X \)-levels, we have \( P_Y \) corresponding to \( \{ \alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7 \} \subseteq \Pi(Y) \). By Lemmas 2.4 and 2.9 again, we have \( \dim(V^2(Q_Y)) \leq \dim(V^2(Q_X)) \leq 6 \cdot \dim(V^1(Q_X)) = 6 \cdot \dim(V^1(Q_Y)) = 96 \). But \( a_4 \neq 0 \), so \( \lambda - \alpha_4 \) is a high weight in \( V^2(Q_Y) \), giving an \( L'_Y \)-module of dimension \( \binom{4}{3}^2 = 100 \). This is a contradiction, so we get no examples in this \( X = D_4 \), \( \delta = \delta_4 \) case.

If \( m = 4 \) and \( \delta \gg \delta_4 \), \( \delta \neq \delta_4 \), then \( \delta \gg \delta_1 + \delta_3 \), which has at its middle level (level 6) more than three weights \( (\delta - 2\beta_1 - 2\beta_2 - 2\beta_3 - 2\beta_4 - 2\beta_5 - 2\beta_6 - 3 \beta_7) \), and \( \delta - \beta_1 - 2\beta_2 - 3 \beta_3 - 2 \beta_4 \), for example). So we in fact get no examples for \( m = 4 \), \( \delta \gg \delta_4 \).

For \( m = 5 \) and \( \delta = \delta_5 \), we have \( Y \) of type \( A_{15} \), and by embedding \( B_X \) into a parabolic subgroup \( P_Y = Q_Y L_Y \) of \( Y \) via \( Q_X \)-levels, we obtain the subgroup corresponding to \( \{ \alpha_4, \alpha_5, \alpha_6, \alpha_{10}, \alpha_{12} \} \subseteq \Pi(Y) \). Here \( \dim(V^2(Q_Y)) \leq 5 \cdot \dim(V^1(Q_Y)) = 10 \) by Lemmas 2.4 and 2.9, which implies \( a_3 = a_5 = a_7 = a_{14} = 0 \). We know \( a_4 = a_6 = 0 \) and the \( a_i \) are symmetric (\( a_i = a_{16-i} \)).
Now we embed the parabolic subgroup of \( X \) corresponding to \( \{ \beta_3, \beta_4, \beta_5 \} \subseteq \Pi(X) \). Then \( P_\gamma = Q_\gamma L_\gamma \) corresponds to \( \Pi(Y) - \{ \alpha_4, \alpha_5, \alpha_1 \} \), so \( L_\gamma' \) is a product of four groups of type \( A_3 \). Since \( a_1, a_2, \alpha_8, a_{14}, \) and \( a_{15} \) are the only possible non-zero coefficients of \( \lambda \), \( \dim(V/[V,Q_\gamma]) = (\dim(V_{A_3})(a_1 \gamma_1 + a_2 \gamma_2))^2 \) (where the \( \gamma_i \) are the fundamental dominant weights of \( A_3 \)). By Lemmas 2.4 and 2.9, we have \( \dim(V^2(Q_\gamma)) \leq 6 \dim(V^1(Q_\gamma)) \), but \( \lambda - \alpha_8 \) is a high weight in \( V^2(Q_\gamma) \), giving an \( L_\gamma' \)-module of dimension \( 4 \cdot 4 \cdot \dim(V^1(Q_\gamma)) \), which is a contradiction. So we have no examples for \( m = 5 \), \( \delta = \delta_5 \).

If \( m = 5 \) and \( \delta \succ \delta_5 \), \( \delta \not\succ \delta_5 \), then \( \delta \succ \delta_1 + \delta_4 \), which has more than three weights at its middle level as in the \( m = 4 \) case above. So there are no examples for \( m = 5 \), \( \delta \succ \delta_5 \).

This completes the argument for \( X = D_m \).

5.3. \( X = E_6 \). Assume \( X = E_6 \). We again need a few “small” weights, among which every dominant \( E_6 \)-weight has a subdominant weight.

**Lemma 5.6.** If \( \delta = d_1 \delta_1 + d_2 \delta_2 + d_3 \delta_3 + d_4 \delta_4 + d_5 \delta_5 + d_6 \delta_6 \) is a non-zero dominant weight of \( T_X \), then \( \delta \) has one of \( \delta_1, \delta_6 \), or \( 0 \) as a subdominant weight.

**Proof.** See problem III.13 in [7]. \( \square \)

Assume \( \delta \succ 0 \). We know \( \delta \neq 0 \) as \( X \) acts irreducibly on \( W \); adding positive roots to 0, the unique lowest dominant weight in the order is \( \delta_2 \). Continuing to add roots, the unique successor to \( \delta_2 \) in the partial order on dominant weights is \( \delta_1 + \delta_6 \). As \( \delta \neq \delta_2 \) (since \( t \) does not act on \( W \)), we have \( \delta \succ \delta_1 + \delta_6 \). The weight \( \delta_1 + \delta_6 \) has 0 as a weight at level 16; a simple check show that there are at least two other weights at this level. So if \( \delta \succ \delta_5 \) we have no examples.

If \( \delta \succ \delta_1 \) (\( \delta \succ \delta_6 \) is the same argument by symmetry), then \( \delta \) has at least three weights at its middle level because \( \delta_1 \) does. The middle level for \( \delta_1 \) is 8 (as the low weight is at level 16), and at level 8 are the three weights

\[
\begin{align*}
\delta_1 &- \beta_1 - \beta_2 - 2\beta_3 - 2\beta_4 - \beta_5 - \beta_6 = \delta_1 - \delta_3 + \delta_5 - \delta_6, \\
\delta_1 &- \beta_1 - \beta_2 - \beta_3 - 2\beta_4 - 2\beta_5 - \beta_6 = \delta_1 - \delta_5, \text{ and} \\
\delta &- 2\beta_1 - 2\beta_3 - 2\beta_4 - \beta_5 = -\delta_1 + \delta_6.
\end{align*}
\]

So there are no examples for \( X = E_6 \).

This completes the proof of Theorem 8.1. \( \square \)

What remains to be considered to complete the proof of Theorem 1 are the cases in which \( X \) acts irreducibly on \( W \), and \( W \) is \( t \)-stable (if \( t \in G \)). This will be completed in part II ([5]).
TABLE 1. Examples arising from the connected case

<table>
<thead>
<tr>
<th>No.</th>
<th>X</th>
<th>Y</th>
<th>$W \mid X$</th>
<th>$V \mid X$</th>
<th>$V \mid Y$</th>
<th>char($K$)</th>
</tr>
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<tbody>
<tr>
<td>I4</td>
<td>$D_n$</td>
<td>$A_{2n-1}$</td>
<td>$\delta_1$</td>
<td>$\ldots$</td>
<td>$1$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(n $\geq$ 4)</td>
<td>$\beta_k$</td>
<td>$n - 1 &gt; k \geq 2$</td>
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</tr>
<tr>
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<td>$D_n$</td>
<td>$A_{2n-1}$</td>
<td>$\delta_1$</td>
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<td>$1$</td>
<td>$\ldots$</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>(n $\geq$ 4)</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>I6</td>
<td>$A_3$</td>
<td>$A_5$</td>
<td>$\delta_2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>II1</td>
<td>$A_5$</td>
<td>$C_{10}$</td>
<td>$\delta_3$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
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<td>$A_2$</td>
<td>$B_3$</td>
<td>$\delta_1 + \delta_2$</td>
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<td>2</td>
<td>2</td>
</tr>
<tr>
<td>S7</td>
<td>$A_3$</td>
<td>$D_7$</td>
<td>$\delta_1 + \delta_3$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>S8</td>
<td>$D_4$</td>
<td>$D_{13}$</td>
<td>$\delta_2$</td>
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<td>1</td>
<td>1</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>($s, t \in Y$)</td>
<td></td>
<td></td>
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<tr>
<td>MR4</td>
<td>$D_n$</td>
<td>$C_n$</td>
<td>$\delta_1$</td>
<td>$\ldots$</td>
<td>$c_1$</td>
<td>$c_2$</td>
</tr>
</tbody>
</table>
### Table 2. New Examples

| No. | X      | Y      | W|X | V|X | V|Y | char(K) |
|-----|--------|--------|---|---|---|---|---|---------|
| U1  | $A_{n-1}$ | $A_n$  | (n odd) | usual | $\bullet \cdots \beta_{n+1} \cdots \bullet$ | $\bullet \cdots \alpha_{n+1} \cdots \bullet$ | any |
| U2  | $D_n$   | $B_n$  | usual | $a_1 \cdots a_n-2 a_n^{-1}$ | $a_1 \cdots a_n^{-1} 1$ | below | $a_n+1$ |
|     |         |        |       | with $a_i + a_j \equiv i - j \pmod{p}$ whenever $a_i$ and $a_j$ are non-zero coefficients with only 0's between them and $i < j < n$; and $2a_i \equiv -2(n-i) - 1 \pmod{p}$ for $a_i$ the last non-zero coefficient before $a_n = 1$. |
| U3  | $D_n$   | $D_{n+1}$ | usual | $\bullet \cdots$ | $\bullet \cdots$ | any |
| U4  | $A_3$   | $D_4$  | $\delta_1 \oplus \delta_3$ | 1 | any |
| U5  | $A_3$   | $D_{10}$ | 2$\delta_2$ | 3 | 1 | 1 | 1 | $p \neq 2,3,5,7$ |
| U6  | $D_m$   | $C_m, B_m$ | $\delta_1$ | 1 | any |
| U7  | $A_2$   | $A_5$  | 2$\delta_1$ | 3 | 1 | 1 | 1 | $p \neq 2,3$ |
| U8  | $A_3$   | $A_9$  | 2$\delta_1$ | 2 | 2 | 1 | 1 | $p \neq 2,5$ |
| U9  | $A_4$   | $A_9$  | $\delta_2$ | 2 | 1 | 1 | 1 | $p \neq 2,5$ |

### References


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