Homework #1

Please provide answers to the following questions in the space provided (i.e., print out and hand in these pages). Also, you must provide all work on separate sheets of paper. Be sure to show your work and to clearly label your graphs. Note: You are welcome to work together on these problems. However, you must show, and turn in, your own work. Also, I highly recommend attempting the problems on your own prior to seeking assistance.

1. **Differentiation** Calculate the derivative of the following functions and find the slope of the functions at \( x_0 = 4 \):

   a) \( f(x) = 3x^{25} \)
   
   \[ f'(x) = m_{x_0=4} = \]

   b) \( f(x) = \frac{10}{x^4} \)
   
   \[ f'(x) = m_{x_0=4} = \]

   c) \( f(x) = \frac{10x}{x^4} \)
   
   \[ f'(x) = m_{x_0=4} = \]

   d) \( f(x) = \sqrt{x} \)
   
   \[ f'(x) = m_{x_0=4} = \]

   e) \( f(x) = \frac{2x^2}{\sqrt{x}} \)
   
   \[ f'(x) = m_{x_0=4} = \]

   f) \( f(x) = 10x - x^2 \)
   
   \[ f'(x) = m_{x_0=4} = \]

   g) \( f(x) = (10 - x)^2 \)
   
   \[ f'(x) = m_{x_0=4} = \]

   h) Use the *difference quotient* to derive \( f'(x) \) given \( f(x) = 10 + 3x \)
   
   \[ f'(x) = \]

2. **Profit Maximization.** Provide the answers to the following questions on a separate sheet of paper. Fancy Fresh Recycling (FFR) company has monopoly power in its region. However, they are not very analytically adept so, heretofore, they have determined their output \((Q)\), price \((P)\) combination using the chicken method. In a moment of lucidity, the owner decided to ditch the old voodoo for more modern voodoo, i.e., she wants to hire an economist. She hires you for your expert opinion on the output and price that will leave the firm best-off. After rigorous analysis, you determine the regional (inverse) demand for recycling services to be: \( P(Q) = 10 - \frac{1}{2}Q \). In addition, based on past company data, you determine the the firms total cost structure can be approximated by the function: \( TC(Q) = 2Q + 3Q^2 \). Using this information, and your knowledge that profit \((\pi)\) is: \( \pi(Q) = TR(Q) - TC(Q) \), you can answer the owners following questions (suppose that \( Q \) represents tons of recycling and the relevant time period is one week),

   a) The owner runs many other businesses and has a hard time keeping track of money. She wants to know if the current output of 5 tons per week is the best that FFR can do?

   b) Explain how you would determine the optimal number of tons of recycling per week for FFR. Give an explanation of how this relates to the optimal decision rule: \( MB = MC \).

3. **Shapes of Functions.** In economics (and many other disciplines) we are not only interested in whether a function is increasing or decreasing (i.e., the slope) at a specific point or along an interval. Interest also lies in the general trend in how the slope is changing around a point or along an interval (i.e., is the slope increasing or decreasing as we move away from, say, \( x_0 \)).
In other words, we are interested in the slope and the rate at which the slope is changing. These two characteristics of a function describes the general shape of a function at a point of along an interval. For instance a firms decision on whether or not to increase output is going to depend, in part, on whether their total costs are: increasing at a decreasing rate (i.e., they are experiencing increasing returns to their variable inputs), or increasing at an increasing rate (i.e., they are experiencing decreasing returns to their variable inputs).

We already know that the derivative of of a function evaluated at a specific point is equal to the slope of the function at that point: \( f'(x) = m \). The rate at which that slope \( m \) is changing (i.e., whether the function is getting steeper or flattening out) is determined by taking the derivative of the original derivative function (i.e., the second derivative): \( f''(x) = \frac{d}{dx} f'(x) \).

Using our in-class example:

\[
f(x) = 10x - x^2.
\]

We know the first derivative:

\[
f'(x) = 10 - 2x.
\]

If we simply take the derivative (with respect to \( x \)) again:

\[
\frac{d}{dx} f'(x) = -2,
\]

we have calculated the second derivative of \( f(x) \). We are simply going to be interested in if the second derivative is positive or negative when evaluated at \( x_0 \). Some rules that we will use:

1) If \( f'(x) > 0 \) \( \Rightarrow \) function is increasing (positive slope).
2) If \( f'(x) < 0 \) \( \Rightarrow \) function is decreasing (negative slope).
3) If \( f''(x) > 0 \) \( \Rightarrow \) function is convex (U-shaped along the range).
4) If \( f''(x) < 0 \) \( \Rightarrow \) function is concave (U-shaped along the range).

For instance, given \( f(x) = 10x - 2x \), \( f'(x) = 10 - 2x \) and \( f''(x) = -2 \), we can say that at \( x_0 = 2 \), the function is increasing (because the slope \([6]\)] is positive) at a decreasing rate. So taking the first and second derivatives of a function in tandem, we can state these final rules:

1’) If \( f'(x) > 0 \) and \( f''(x) > 0 \) \( \Rightarrow \) function is increasing at an increasing rate.
2’) If \( f'(x) > 0 \) and \( f''(x) < 0 \) \( \Rightarrow \) function is increasing at a decreasing rate.
3’) If \( f'(x) < 0 \) and \( f''(x) > 0 \) \( \Rightarrow \) function is decreasing at an increasing rate.
4’) If \( f'(x) < 0 \) and \( f''(x) < 0 \) \( \Rightarrow \) function is decreasing at a decreasing rate.

\[1\] A function is convex along a range if the secant line along the same range lies above the graph or, conversely, the tangent line lies below the graph.

\[2\] A function is concave along a range if the secant line along the same range lies below the graph or, conversely, the tangent line lies above the graph.

\[3\] Note that because this function is a parabola, the slope is always decreasing as we increase \( x \). In other words the second derivative is not a function of \( x \). However, as we will see, some functions can change from concave to convex (or vice versa) along ranges of \( x \).
Using rules 1-4 and 1'-4', determine: (1) the slope at $x_0$ and $x_1$, (2) if the function convex or concave between $x_0$ and $x_1$, (3) a sketched graph of the function (on a separate sheet) along the range $(x_0 - x_1)$ (i.e., determine the value of $f(x)$ at $x_0$ and $x_1$ and the general shape of the function along that range), for the following functions and values of $x$:

<table>
<thead>
<tr>
<th>Function</th>
<th>$x_0$</th>
<th>$x_1$</th>
<th>$m_{x_0}$</th>
<th>$m_{x_1}$</th>
<th>Convex/Concave</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) $f(x) = x^2$, $x_0 = 1$, $x_1 = 3$.</td>
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<td></td>
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<tr>
<td>b) $f(x) = 1/x$, $x_0 = 1/2$, $x_1 = 3$.</td>
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<tr>
<td>c) $f(x) = 10x^2 - \frac{2}{3}x^3$, $x_0 = 0$, $x_1 = 2$.</td>
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<tr>
<td>c) $f(x) = 10x^2 - \frac{2}{3}x^3$, $x_0 = 3$, $x_1 = 5$.</td>
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4. **Increasing and Decreasing Returns.** Provide the answers to the following questions on a separate sheet of paper. Jimbob’s Pizza Shack currently employs one person, Jimbob. Jimbob hires you to advise him on the prudence of hiring additional employees. You think that a good indicator would be to determine where he currently lies on his production function. Based on your extensive knowledge of pizza parlors of similar size, etc., you estimate his production function (as a function of labor ($l$), given his current level of capital) to be: $f(l) = -l + 12l^2 - 2l^3$. Keeping in mind that when the marginal product of labor (MPL, i.e., the marginal increase in output associated with increasing labor) is positive and increasing at an *increasing* rate (think of rules 1'-4') a firm is experiencing increasing returns to labor, when the MPL is positive but increasing at a *decreasing* rate a firm is experiencing decreasing returns to labor, answer the following questions:

a) At his current level of employment, is Jimbob experiencing increasing or decreasing returns?

b) How would you characterize Jimbob’s returns to labor if he were to add 1, 2, 3, or 4 more employees?

c) Can you determine the specific point (number of employees at which the returns to labor changes? (*Hint:* Using the above rules, think about how you would determine where a function changes from concave to convex and vice versa.)

*Hint 2:* It may be helpful to use a graphing utility (such as [www.fooplot.com](http://www.fooplot.com)) to visualize the function. Also, I recommend setting the range of the window to $0 < x < 6$ and $0 < y < 100$. 

Hanauer 305, Spring 2013