1. (a) By substituting appropriate values into the binomial theorem, find a formula for the sum

\[ \binom{n}{0} + 3 \binom{n}{1} + 3^2 \binom{n}{2} + 3^3 \binom{n}{3} + \cdots + 3^n \binom{n}{n}. \]

(b) Give a combinatorial proof of the formula you found in part (a). (Hint: Count the number of ways to paint \( n \) different houses either red, blue, green, or not at all.)

Solution:

(a) Letting \( x = 1 \) and \( y = 3 \) in the binomial theorem, we have

\[ 4^n = (1 + 3)^n = \binom{n}{0}1^n + \binom{n}{1}1^{n-1} \cdot 3 + \binom{n}{2}1^{n-2} \cdot 3^2 + \binom{n}{3}1^{n-3} \cdot 3^3 + \cdots + \binom{n}{n}1^0 \cdot 3^n \]

so our formula for the sum is \( 4^n \).

(b) The formula that we are trying to prove is

\[ \binom{n}{0} + 3 \binom{n}{1} + 3^2 \binom{n}{2} + 3^3 \binom{n}{3} + \cdots + 3^n \binom{n}{n} = 4^n. \]

We claim that both sides of (*) count the number of ways to paint \( n \) distinct houses either red, blue, green or not at all, which we verify below:

\textbf{Left-hand side:} Here, we enumerate the number of different ways to paint by using cases according to the number of houses we paint.

\textbf{Case 1.} (No houses get painted.) In this case, we are effectively choosing 0 houses to paint, which can be done in \( C(n, 0) \) ways.

\textbf{Case 2.} (1 house gets painted.) We break this task into two parts: pick 1 house to be painted in \( C(n, 1) \) ways, and then choose to paint it either red, green, or blue in 3 ways.

\textbf{Case 3.} (2 houses get painted.) Again, break this task into two parts: pick 2 houses to be painted in \( C(n, 2) \) ways, and then choose colors for both houses in \( 3^2 \) ways.

\[ \vdots \]

\textbf{Case n.} (\( n \) houses get painted.) Pick all \( n \) houses to be painted in \( C(n, n) \) ways, and then choose colors for all \( n \) houses in \( 3^n \) ways.

Summing over all the cases above accounts for all ways of painting and gives the left-hand side of (*).

\textbf{Right-hand side:} Here, we enumerate the number of different ways to paint by simply observing that for each house, we have four choices: either paint it red, blue, green, or don’t paint it at all. Therefore, there are \( 4^n \) total ways to paint, which gives the right-hand side of (*).

2. How many 10-letter “words” from the standard 26-letter English alphabet contain each of the letters \( a, b, c, \) and \( d \) at least once?

Solution: Since it is easier to count the number of words that are missing a particular letter than those that have it, we begin by defining the following sets:

\[ S = \{ \text{10-letter “words”} \} \]
Above are solutions to Sample Exam 2 Problems – Math 316.

Using Inclusion/Exclusion, our answer will then be

\[ |S| - |A_1 \cup A_2 \cup A_3 \cup A_4| = 26^{10} - |A_1 \cup A_2 \cup A_3 \cup A_4| \]

\[ = 26^{10} - |A_1| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + |A_1 \cap A_2 \cap A_3 \cap A_4| \]

We now examine each of the sums separately, noting that each sum represents the number of words missing a particular number of letters:

**Words in \( \sum |A_i| \): that is, words missing one letter:**
- Choose a value of \( i \) \( \leftarrow \) \( C(4, 1) \) ways
  (Here, we are essentially choosing one of the four letters, \( a, b, c, \text{ or } d \), for our word to be missing.)
- Choose letters for our 10 positions \( \leftarrow \) \( 25^{10} \) ways
  (Note that, since each word is missing one letter, there are only 25 choices for each of our 10 positions.)

**Words in \( \sum |A_i \cap A_j| \): that is, words missing two letters:**
- Choose a value for \( i \) and \( j \) \( \leftarrow \) \( C(4, 2) \) ways
  (Here, we are essentially choosing two of the four letters, \( a, b, c, \text{ or } d \), for our word to be missing.)
- Choose letters for our 10 positions \( \leftarrow \) \( 24^{10} \) ways
  (Note that, since each word is missing two letters, there are only 24 choices for each of our 10 positions.)

**Words in \( \sum |A_i \cap A_j \cap A_k| \):**
- Choose a value for \( i, j, \text{ and } k \) \( \leftarrow \) \( C(4, 3) \) ways
- Choose letters for our 10 positions \( \leftarrow \) \( 23^{10} \) ways

**Words in \( \sum |A_i \cap A_j \cap A_k \cap A_p| \):**
- Choose a value for \( i, j, k, \text{ and } p \) \( \leftarrow \) \( C(4, 4) = 1 \) way
- Choose letters for our 10 positions \( \leftarrow \) \( 22^{10} \) ways

To summarize, we see that there are \( C(4, 1) \cdot 25^{10} \) words that are missing exactly one of the letters \( a, b, c, \text{ or } d \); there are \( C(4, 2) \cdot 24^{10} \) words missing exactly two of the letters; there are \( C(4, 3) \cdot 23^{10} \) words that are missing exactly three of the letters, and \( C(4, 4) \cdot 22^{10} \) words missing all four of the letters. Referring to the equation above, our final answer is therefore

\[ 26^{10} - \binom{4}{1} 25^{10} + \binom{4}{2} 24^{10} - \binom{4}{3} 23^{10} + \binom{4}{4} 22^{10} . \]

3. Solve the following recurrence relations.

(a) \( h_n = -h_{n-1} + 6h_{n-2} \quad (n \geq 2), \quad h_0 = 7, \quad h_1 = -11 \)

(b) \( h_n = 5h_{n-1} - 8h_{n-2} + 4h_{n-3} \quad (n \geq 3), \quad h_0 = -2, \quad h_1 = 1, \quad h_2 = 13 \)

**Solution:**
(a) First, note that the characteristic equation of this recurrence is given by \( x^2 + x - 6 = 0 \), so we have

\[ x^2 + x - 6 = 0 \implies (x + 3)(x - 2) = 0, \]

which yields a general solution of

\[ h_n = a \cdot 2^n + b \cdot (-3)^n. \]

The initial conditions \( h_0 = 7 \) and \( h_1 = -11 \) lead to the equations \( a + b = 7 \) and \( 2a - 3b = -11 \), respectively. Solving for \( b \) in the first equation, we get \( b = 7 - a \), and substituting into the second equation, we obtain

\[ 2a - 3(7 - a) = -11 \implies 2a - 21 + 3a = 11 \implies 5a = 10, \]

so we conclude that \( a = 2 \) and \( b = 7 - a = 5 \). Our final answer is therefore

\[ h_n = 2 \cdot 2^n + 5 \cdot (-3)^n. \]

(b) First, note that the characteristic equation of this recurrence is given by \( x^3 - 5x^2 + 8x - 4 = 0 \), so we have

\[ x^3 - 5x^2 + 8x - 4 = (x - 1)(x^2 - 4x + 4) = (x - 1)(x - 2)^2 = 0, \]

yielding a repeated characteristic root of 2 and a single root of 1. Our general solution is therefore given by

\[ h_n = a \cdot 2^n + bn \cdot 2^n + c \cdot 1^n = a \cdot 2^n + bn \cdot 2^n + c. \]

Our initial conditions \( h_0 = 2, \ h_1 = 1, \) and \( h_2 = 13 \) lead to the equations \( a + c = -2 \), \( 2a + 2b + c = 1 \), and \( 4a + 8b + c = 13 \), respectively. Using standard elimination techniques, we reduce our system of equations as follows:

\[
\begin{align*}
2a + 2b + c &= 1 \\
4a + 8b + c &= 13
\end{align*}
\rightarrow
\begin{align*}
2b - c &= 5 \\
8b - 3c &= 21
\end{align*}
\rightarrow
\begin{align*}
a + c &= -2 \\
a + c &= -2
\end{align*}
\]

We therefore see that \( c = 1 \), so

\[ 2b = c + 5 \implies b = 3, \quad \text{and} \quad a = -2 - c = 3, \]

which yields a final answer of

\[ h_n = -3 \cdot 2^n + 3n \cdot 2^n + 1 = 2^n(3n - 3) + 1. \]

4. A donut shop sells 30 different kinds of donuts, including chocolate donuts, powdered donuts, jelly donuts, and 27 other types.

(a) Find a simplified generating function for the number of unordered selections of \( n \) donuts if

i. there are no restrictions.

ii. the selection must contain at least 5 chocolate donuts.

iii. the selection must contain at least 1 chocolate, at least 1 powdered, but no more than 3 jelly.

(b) Sam wants to purchase 12 donuts for a wild pastry party at Dr. Luttmann’s house. Unfortunately, the donut shop is out of chocolate donuts, but it is offering a special sale on jelly donuts: three for $1. Use generating functions to find the number of different donut selections Sam can make assuming that he wants to buy a multiple of 3 number of jelly donuts.

Solution:

(a) For each of the following functions, recall that the exponent on the variable \( x \) determines the number of donut selections made.
i. Since we can choose any number of each type of donut, our generating function is given by

\[
f(x) = (1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots) \cdots (1 + x + x^2 + \cdots)
\]

so our answer is \( f(x) = \frac{1}{(1-x)^{30}} \).

ii. Since we want to have at least 5 chocolate donuts, the “chocolate factor” of our function will look like \((x^5 + x^6 + x^7 + \cdots)\), while the factors for the other 29 donut varieties will stay the same as in part (i). Therefore, we have

\[
f(x) = (x^5 + x^6 + x^7 + \cdots)(1 + x + x^2 + \cdots)^{29}
\]

so our final answer is \( f(x) = \frac{x^5}{(1-x)^{30}} \).

iii. In this case, we have

\[
f(x) = (x + x^2 + x^3 + \cdots)(x + x^2 + x^3 + \cdots) \cdots (x + x^2 + x^3 + \cdots)^{29}
\]

which yields a final answer of \( f(x) = \frac{x^2(1-x^4)}{(1-x)^{30}} \).

(b) First, we find a generating function for \( d_n \), the number of selections of \( n \) donuts in which no chocolate donuts are chosen, and a multiple of 3 jelly donuts are chosen. Reasoning as above, our generating function looks like

\[
f(x) = (1 + x^3 + x^6 + \cdots)(1 + x + x^2 + \cdots)^{28}
\]

so our answer is

\[
f(x) = \frac{1}{1-x^3} \cdot \frac{1}{(1-x)^{28}} = \frac{1}{(1-x^3)(1-x)^{28}}.
\]

Now, we can either use technology to find the Taylor series for \( f \) and read off the coefficient of the \( "x^{12n}" \) term, or we can analyze algebraically as follows:

\[
f(x) = (1 + x^3 + x^6 + \cdots) \cdot \frac{1}{(1-x)^{28}}
\]

\[
= (1 + x^3 + x^6 + \cdots) \sum_{k=0}^{\infty} \binom{27+k}{k} x^k
\]

\[
= \sum_{k=0}^{\infty} \binom{27+k}{k} x^k + \sum_{k=0}^{\infty} \binom{27+k}{k} x^{k+3} + \sum_{k=0}^{\infty} \binom{27+k}{k} x^{k+6} + \cdots
\]

We see from the above sum that we get terms with \( x^{12} \) from the 1st sum with \( k = 12 \), from the 2nd sum with \( k = 9 \), from the 3rd sum with \( k = 6 \), etc., which gives a final answer of

\[
h_{12} = \binom{39}{12} + \binom{36}{9} + \binom{33}{6} + \binom{30}{3} + \binom{27}{0} = \sum_{j=0}^{4} \binom{27 + 3j}{3j}
\]
Recall the Fibonacci sequence, which is defined by \( f_0 = 0 \), \( f_1 = 1 \), and \( f_n = f_{n-1} + f_{n-2} \) for \( n \geq 2 
\)
(a) By repeated use of the Fibonacci recursion formula, prove that \( f_n = 5f_{n-4} + 3f_{n-5} \) for all \( n \geq 5 \).
(b) Use part (a) to help you prove that \( f_{5n} \) is a multiple of 5 for all \( n \geq 0 \).

**Solution:**

(a) Consider any \( n \geq 5 \). Then we have

\[
\begin{align*}
  f_n &= f_{n-1} + f_{n-2} \\
  &= f_{n-3} + f_{n-2} + f_{n-4} + f_{n-3} \\
  &= f_{n-2} + 2f_{n-3} + f_{n-4} \\
  &= f_{n-4} + f_{n-3} + 2f_{n-3} + f_{n-4} \\
  &= 3f_{n-3} + 2f_{n-4} \\
  &= 3(f_{n-5} + f_{n-4}) + 2f_{n-4} \\
  &= 5f_{n-4} + 3f_{n-5}.
\end{align*}
\]

(b) We will prove this by induction on \( n \). For \( n = 0 \), observe that

\[
f_{5n} = f_0 = 0 = 5 \cdot 0,
\]

which establishes the basis step.

Now, assume that \( f_{5k} \) is a multiple of 5 for some \( k \geq 0 \). Then there exists an integer \( p \) such that \( f_{5k} = 5p \), and we have

\[
\begin{align*}
  f_{5(k+1)} &= f_{5k+5} \\
  &= 5f_{(5k+5)-4} + 3f_{(5k+5)-5} & \text{by part (a)} \\
  &= 5f_{5k+1} + 3f_{5k} \\
  &= 5f_{5k+1} + 3(5p) & \text{induction hypothesis} \\
  &= 5(f_{5k+1} + 3p),
\end{align*}
\]

which shows that \( f_{5(k+1)} \) is a multiple of 5. Therefore, by induction, we conclude that \( f_{5n} \) is a multiple of 5 for all integers \( n \geq 0 \).

A fixed chessboard is to be completely tiled with the two shapes of pieces shown to the right. Assume there is a plentiful supply of white \( 1 \times 1 \) monominoes, and plentiful supplies of red, yellow, and green of the “L-shaped” piece.

(a) Assuming that the configurations \( \square \) and \( \blacksquare \) are not allowed anywhere in the tiling, how many different ways are there to tile a \( 2 \times 200 \) chessboard?

(b) Now, assume that only white monominoes and red L-shaped pieces are available, but all configurations are now allowed, including the two that were prohibited in part (a). Find, but DO NOT SOLVE, a recurrence relation for \( g_n \), the number of ways to tile a \( 2 \times n \) chessboard with these pieces. Then, use the recurrence relation to find the number of tilings of a \( 2 \times 5 \) chessboard.

**Solution:**

(a) Let \( h_n \) represent the number of ways to tile a \( 2 \times n \) chessboard with the previously described pieces. We will begin by finding a recurrence relation for \( h_n \).
Referring to the diagram to the right, we see that there are \( h_{n-1} \) ways to tile the chessboard if the tiling starts with two monominoes in the first column. On the other hand, if the first two columns contain one L-shaped piece and a monomino, there are three possible colors for the L-shaped piece, giving \( 3h_{n-2} \) ways to complete the tiling. But there are 4 different ways to orient the L-shaped piece by rotating it \( 90^\circ \) each time, giving \( 4 \cdot 3h_{n-2} = 12h_{n-2} \) total ways for a \( 2 \times n \) tiling to have one L-shaped piece and one monomino in the first two columns. Our recurrence formula is therefore given by

\[
h_n = h_{n-1} + 12h_{n-2} \quad \text{for } n \geq 3.
\]

To determine the initial conditions, we first note that \( h_1 = 1 \) because there is only one way to tile a \( 2 \times 1 \) chessboard: with two white monominoes. We also observe that \( h_2 = 13 \) by enumerating all of the possibilities below:

Now, we are ready to solve the recurrence relation \( h_n = h_{n-1} + 12h_{n-2} \). Since

\[
x^2 - x - 12 = 0 \implies (x - 4)(x + 3) = 0,
\]

the general solution to our recurrence relation is

\[
h_n = a \cdot 4^n + b \cdot (-3)^n.
\]

The initial conditions \( h_1 = 1 \) and \( h_2 = 13 \) lead to the equations \( 4a - 3b = 1 \) and \( 16a + 9b = 13 \), which when solved yield \( a = 4/7 \) and \( b = 3/7 \). Therefore,

\[
h_n = \frac{4^{n+1} - (-3)^{n+1}}{7} \quad \text{for all } n \geq 1,
\]

and so there are

\[
h_{200} = \frac{4^{201} - (-3)^{201}}{7} = \frac{4^{201} + 3^{201}}{7}
\]

ways to tile a \( 2 \times 200 \) chessboard in the desired manner.

(b) Referring to the diagram to the right, we see that there are several different ways for a tiling of a \( 2 \times n \) chessboard to begin. If a tiling starts with two monominoes in the first column, there are \( g_{n-1} \) ways to complete the tiling. If a tiling does not start with two monominoes, then it could either start with one L-shaped piece and a monomino (\( 4g_{n-2} \) like this) or with two L-shaped pieces (\( 2g_{n-3} \) like this). Our recurrence formula is therefore given by

\[
g_n = g_{n-1} + 4g_{n-2} + 2g_{n-3}
\]

for \( n \geq 4 \). Our goal is to find \( g_5 \).

In order to find \( g_5 \), we need several initial conditions. Clearly, \( g_1 = 1 \), since a \( 2 \times 1 \) chessboard can only be tiled in one way: with 2 monominoes. On the other hand, a \( 2 \times 2 \) chessboard can either be tiled with 4 monominoes, or with any of the 4 configurations shown in the first two columns of the rows labeled
“Type 2” through “Type 5” in the diagram above, so \( g_2 = 5 \). Finally, a 2 \( \times \) 3 chessboard can be tiled with six monominoes (1 way), with the first column of the Type 1 configuration followed by the first two columns of any of the Type 2 through Type 5 configurations (4 ways), with the first two columns of any of the Type 2 through Type 5 configurations followed by the first column of the Type 1 configuration (4 ways), or the first three columns of the Type 6 or Type 7 configurations (2 ways), giving \( g_3 = 1 + 4 + 4 + 2 = 11 \).

Therefore, we have

\[
\begin{align*}
g_4 &= g_3 + 4g_2 + 2g_1 = 11 + 4 \cdot 5 + 2 \cdot 1 = 33 \\
g_5 &= g_4 + 4g_3 + 2g_2 = 33 + 4 \cdot 11 + 2 \cdot 5 = 87,
\end{align*}
\]

so we conclude that there are \( g_5 = 87 \) ways to tile a 2 \( \times \) 5 chessboard with these pieces.

7. For each of the following bipartite graphs, you are given a matching \( M_1 \), indicated by the shaded edges. By finding successive \( M_1 \)-alternating paths (if they exist), arrive at a max-matching \( M^* \), and then show that your final matching is a max-matching by writing down an appropriate cover. Clearly indicate your paths and matchings at each step in the process.

(a) Below is an \( M_1 \)-alternating path \( \gamma_1 \):

\[
\gamma_1 : \ x_4, y_2, x_2, y_3, x_3, y_4
\]

This path leads to the max-matching

\[
M^* = \{\{x_1, y_1\}, \{x_4, y_2\}, \{x_2, y_3\}, \{x_3, y_4\}\},
\]

which is confirmed by observing that the set

\[
S = \{x_1, x_2, x_3, x_4\}
\]

is a cover of the graph with \( |S| = |M^*| = 4 \). The max-matching \( M^* \) and the minimum cover \( S \) are illustrated in the diagram to the right.

(b) Below is an \( M_1 \) alternating path and its induced new matching \( M_2 \) with one more edge (see diagram to the immediate right):

\[
\gamma_1 : \ x_2, y_4, x_4, y_2 \\
M_2 = \{\{x_2, y_4\}, \{x_4, y_2\}\}
\]

Since \( M_2 \) is still not a max-matching, we find an \( M_2 \)-alternating path \( \gamma_2 \) and its induced max-matching \( M^* \) (see diagram above to the far right).

\[
\gamma_2 : \ x_1, y_1 \\
M^* = \{\{x_2, y_4\}, \{x_4, y_2\}, \{x_1, y_1\}\}
\]
To answer your question, yes, $\gamma_2$ is indeed a legal $M_2$ alternating path (check the definition, and note that one of the properties is vacuously satisfied)! The fact that $M^*$ is a max-matching is confirmed by the fact that $S = \{x_1, y_2, y_4\}$ is a cover of the graph with $|S| = |M^*| = 3$.

(c) Below is an $M_1$-alternating path $\gamma_1$:

$$\gamma_1 : \ x_1, y_2, x_2, y_3$$

This path leads to the max-matching

$$M^* = \{\{x_3, y_1\}, \{x_1, y_2\}, \{x_2, y_3\}\},$$

which is confirmed by observing that the set

$$S = \{x_2, x_3, y_2\}$$

is a cover of the graph with $|S| = |M^*| = 3$. The max-matching $M^*$ and the minimum cover $S$ are illustrated in the diagram to the right.

8. In each of the following, let $G$ denote a bipartite graph. Either give an example that satisfies the given conditions, or show why no example exists. If an example exists, make sure to clearly indicate the desired portions of the graph in your example, and to specify why it works.

- (a) A matching $M$ such that $|M| < \rho(G)$.
- (b) A matching $M$ such that $|M| < c(G)$.
- (c) A matching $M$ such that $|M| > c(G)$.
- (d) A cover $S$ such that $|S| > c(G)$.
- (e) A cover $S$ such that $|S| < \rho(G)$.

Solution:

(a) The graph $G_1$ to the right works, because $M = \{\{x_1, y_1\}\}$ is a matching in $G_1$ with fewer edges than the max-matching $M^* = \{\{x_1, y_1\}, \{x_2, y_2\}\}$. Therefore,

$$|M| = 1 < 2 = \rho(G_1).$$

(b) The same graph, $G_1$, and matching, $M$, as in part (a) works as an example for (b) because no single vertex in $G_1$ meets every edge of the graph. Therefore, the minimum size of a cover of $G_1$ is 2, and we have

$$|M| = 1 < 2 = c(G_1).$$

(c) No such example exists, for if it were true that such a matching $M$ existed, we would have

$$\rho(G) \geq |M| \quad \leftarrow \quad \text{by definition of } \rho(G)$$

$$> c(G), \quad \leftarrow \quad \text{by given information}$$

which would contradict the fact that $\rho(G) = c(G)$.

(d) The graph $G_2$ to the right works, because $S = \{x_1, x_2\}$ is a cover of $G_2$, and $S^* = \{y_1\}$ is a minimal cover of $G_2$ because the removal of 0 vertices leaves all edges of $G_2$ in the graph. Therefore, we have

$$|S| = 2 > 1 = c(G_2).$$
(e) No such example exists, for if it were true that such a cover $S$ existed, we would have

$$
c(G) \leq |S| \quad \leftarrow \quad \text{by definition of } c(G)
$$

$$
c(G) < \rho(G), \quad \leftarrow \quad \text{by given information}
$$

which would contradict the fact that $c(G) = \rho(G)$. 