Is There a Formula that Generates Prime Numbers?

A Sonoma State
M*A*T*H Colloquium

Presentation by

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Today Decides Tomorrow

What *is* a prime number?

A positive integer p is called a *prime number* provided p has exactly two positive divisors.

$p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, ...

Is there a formula for primes?

- Is there a function *f* such that $f(n) = p_n$,the *n*-th prime, for all *n* ?
- Is there a function that produces only prime numbers ?
- Are there polynomial functions that produce primes?

Are there formulas for primes?

YES!

Astonishing Formulas

Amusing Formulas

Worthless Formulas

Some Notation

Let n be a positive integer.

Let $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$.

Let $\lfloor x \rfloor$ represent the greatest integer that does not exceed x.

A Tool

$\pi(n)$ = # primes not exceeding n

Find $\pi(n)$ for $n = 1, 2, 3, 4, 5, 10, 20$.

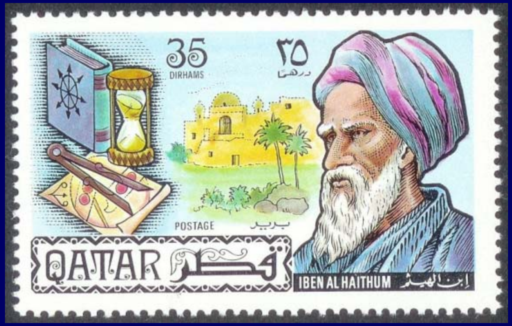
n	1	2	3	4	5	10	20
π(n)	0	1	2	2	3	4	8

A Worthless Formula for p_n

Willans, 1964

$$p_n = 1 + \sum_{k=1}^{2^n} \left\lfloor \sqrt[n]{\frac{n}{1+\pi(k)}} \right\rfloor$$

Find p_3.



How Does It Work?

Wilson's Theorem (1770)

n is prime iff $[(n - 1)! + 1] / n$ is an integer.

Willans used Wilson's Theorem to count primes.

$$\pi(k) = -1 + \sum_{j=1}^{k} \left\lfloor \cos^2 \pi \frac{(j-1)!+1}{j} \right\rfloor$$

A Formula with Encoded Information

1952, Sierpinski's other constant

A = 0.0203000500000070...

$$A = \sum_{k=1}^{\infty} p_k 10^{-2^k} \quad \Rightarrow$$

$$p_n = \left\lfloor 10^{2^n} A \right\rfloor - 10^{2^{n-1}} \left\lfloor 10^{2^{n-1}} A \right\rfloor$$

Are There Functions that Produce only Primes?

We want f so that $f(n)$ is always prime, but not necessarily p_n .

We don't want something like

$$f(n) = 17 \quad \text{for all } n .$$

A Function that Produces only Primes

1951, Wright There exists a real number $\omega \approx 1.9287800$ so that the following function is prime for all n.

$$f(n) = \left\lfloor 2^{2^{2^{\cdot^{\cdot^{\cdot^{2^{\omega}}}}}}} \right\rfloor$$

Another Function that Produces only Primes

1947, Mills There exists a real number $\theta \approx 1.3064$ so that the following function is prime for all n.

$$g(n) = \left\lfloor \theta^{3^n} \right\rfloor$$

How about a Nice Function that Produces Lots of Primes?

1772, Euler's function

$f(n) = n^2 + n + 41$

Theorem There is no non-constant polynomial in one variable with integer coefficients which produces only prime values for integer inputs.

Polynomials that Generate Primes

1971, Matijasevic

There exists a polynomial of degree 37 in 24 variables with integer coefficients such that the set of prime numbers coincides with the set of positive values assumed by the polynomial as the variables range in the set of non-negative integers.

A Polynomial whose Positive Values are Prime

1976 Jones, Sato, Wada, Wiens

found an explicit polynomial of degree 25 in 26 variables

$$
\begin{aligned}
&(k+2)\Big\{1-[wz+h+j-q]^2-\big[(gk+2g+k+1)(h+j)+h-z\big]^2\\
&-[2n+p+q+z-e]^2-\big[16(k+1)^3(k+2)(n-1)^2+1-f^2\big]^2\\
&-\big[e^3(e+2)(a+1)^2+1-o^2\big]^2-\big[(a^2-1)y^2+1-x^2\big]^2\\
&-\big[16r^2y^4(a^2-1)+1-u^2\big]^2-[n+l+v-y]^2\\
&-\Big[\big((a+u^2(u^2-a))^2-1\big)(n+4dy)^2+1-(x-cu)^2\Big]^2\\
&-\big[(a^2-1)l^2+1-m^2\big]^2-[ai+k+1-l-i]^2\\
&-\big[p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m\big]^2\\
&-\big[q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x\big]^2\\
&-\big[z+pl(a-p)+t(2ap-p^2-1)-pm\big]^2\Big\}
\end{aligned}
$$

Is There a Formula that Generates Prime Numbers?

Yes, there are many such formulas, but they all seem to be worthless.

References

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References

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