Chapter 6

Probability and Random Processes
Random Experiment

- The fundamental concept in probability theory is the concept of **random experiment**, which is any experiment whose outcome cannot be predicted with certainty.

- A simple example is coin tossing experiment. We know that heads and tails are possible outcomes, although the outcome (head or tail?) of a particular experiment (toss) is uncertain.
A General Communication System

- **Source**: Speech, Video, etc.
- **Transmitter**: Conveys information
- **Channel**: Invariably distorts signals
- **Receiver**: Extracts information signal
- **User**: Utilizes information
Why Learn about Probability Theory?

- What is the probability that this is 1?

Optimum (Correlation) Receiver:
Let us define the following concepts associated with a random experiment:

- **Outcome** ($\xi$) – the result of a random experiment
- **Sample space** ($\Omega$) – the set of all possible outcomes of a random experiment
- **Event** ($A$) – any collection of outcomes, in other words, a subset of $\Omega$
- The empty subset $\emptyset$, is called the **null or impossible event**, and the whole set $\Omega$ is called the **whole or sure event**

**Example: Roll a dice**
- Outcomes: landing with a 1, 2, 3, 4, 5, or 6 face up.
- Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$
- Event: outcome is larger than 4
- Frequency of 1 happening $= \frac{10}{60} = \frac{1}{6}$ (10 occurrence; 60 trials)
- We obtain Probability or Likelihood $\rightarrow$ We try INFINIT times!
In the axiomatic approach, the probability is defined as a function that assigns a real number, denoted by \( P(A) \), to every event \( A \) in the sample space \( \Omega \) such that:

**P1** \( 0 \leq P(A) \leq 1 \)

**P2** The whole event \( \Omega \) will occur each time we perform the random experiment

\[ P(\Omega) = 1 \]

**P3** If the events are **mutually exclusive** (i.e., cannot occur at the same time), the probability of their union is the sum of their probabilities

\[ P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots \]
Probability Axioms

- By using the above axioms, we can derive following important properties of the probability function:

P4 The probability of the null event is zero.

\[ P(\emptyset) = 0 \]

P5 \[ P(\overline{A}) = 1 - P(A), \quad \overline{A} = \text{complement of } A \]

- If the events \( A_1, A_2, \ldots \) are not mutually exclusive, the probability of their union is upper-bounded by the sum of probabilities of the constituent events. That is,

\[ P(A_1 \cup A_2 \cup \ldots) \leq P(A_1) + P(A_2) + \ldots \quad \text{Union Bound} \]
Example

- Rolling a dice. $S = \{1, 2, 3, 4, 5, 6\}$
- Find intersection and union of two events $A$ and $B$

  - Defining Events: Let $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$

  - Union of sets: $A \cup B = \{1, 2, 3, 5\}$

  - Intersection: $A \cap B = \{1, 3\}$

  - $A' = \{4, 5, 6\}$
Example of Union and Intersection

- A card is drawn from a well-shuffled deck of 52 playing cards. What is the probability that it is a queen or a heart?

\( Q = \text{Queen and } H = \text{Heart} \)

\[ P(Q) = \frac{4}{52}, \quad P(H) = \frac{13}{52}, \quad P(Q \cap H) = \frac{1}{52} \]

\[
P(Q \cup H) = P(Q) + P(H) - P(Q \cap H)
\]

\[
= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}
\]
Conditional Probability

- The probability $P(A)$ is *a priori* probability of the occurrence of an event $A$
  - Reflects our knowledge of $A$ before the random experiment takes place
- The **conditional probability** $P(A|B)$ is the *a posteriori* probability of event $A$ knowing that event $B$ has already occurred
- It is defined as
  $$P(A|B) = \frac{P(AB)}{P(B)}$$
  provided $P(B) > 0$
- Conditioning by event $B$ has the effect of restricting the universe of outcomes for the event $A$ to the subset $B$ of $\Omega$

Note: We are assuming $A$ and $B$ are not independent!
Independent Events

- $A$ and $B$ are said to be independent events if
  \[ P(AB) = P(A)P(B) \]

- One should not confuse independent events with mutually exclusive or disjoint events
  - Mutually exclusive events have no outcome in common, i.e., $AB = \emptyset$ implying that $P(AB) = 0$
  - Independent events in most cases are not disjoint

- Substituting into the definition of conditional probability yields
  \[ P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A) \]

- $\Rightarrow$ that the occurrence of $B$ does not provide any more information about the event $A$
Example

• 1A & 1B
• 1C
Rule (Law) of Total Probability

Basically: we can calculate the probability of an event based on other events.

\[ p(A) = \sum P(B_i)P(A \mid B_i) \]
Bayes’ Theorem (simple version)

Theorem (Bayes’ Theorem)

\[ P(A|B) = \frac{P(B|A)P(A)}{P(B)} \]

This lets us express the probability of A given B, in terms of the probability of B given A.

Alternate formulation of Bayes’ Theorem

\[ P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} \]

where we used

\[ P(B) = P(B \cap A) + P(B \cap A^c) = P(B|A)P(A) + P(B|A^c)P(A^c) \]
Full version of Bayes’ Theorem

Definition (Partition of $S$)

Events $A_1, \ldots, A_n$ **partition** the sample space $S$ when

- $S = A_1 \cup \cdots \cup A_n$.
- $A_i \cap A_j = \emptyset$ for $i \neq j$. (**pairwise mutually exclusive**)
- $P(A_i) > 0$ for all $i$.

In other words, $A_1, \ldots, A_n$ are all nonempty with positive probability, and every element of the sample space is in exactly one of them.

Theorem (Bayes’ Theorem)

*Let* $A_1, \ldots, A_n$ *be mutually exclusive events that partition sample space* $S$, *and B be any event on S*. Then

- $P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$
- If $P(B) > 0$ then for each $j = 1, \ldots, n$,

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$$

Can you prove this?

Example

- 1D
- 1E
Example of Conditional Probability

Given:

- \( P_{01} = 0.01 \Rightarrow P_{00} = 1 - P_{01} = 0.99 \)
- \( P_{10} = 0.01 \Rightarrow P_{11} = 1 - P_{10} = 0.99 \)

\[
\Pr(e) = \Pr(0) \cdot P_{01} + \Pr(1) \cdot P_{10} = \frac{1}{2} \cdot 0.01 + \frac{1}{2} \cdot 0.01
\]

\[= 0.01\]
A random variable is defined as a rule that assigns a real number to each possible outcome $\xi \in \Omega$ of a random experiment.

Thus, random variable is a function that maps every outcome $\xi \in \Omega$ to a real number $x$ as illustrated in Figure.

We will denote random variables in a bold font ($x, y, \ldots$) and the values assumed by them are displayed by the lowercase letters ($x, y, \ldots$).
Discrete Random Variables

- Random variables may be discrete, continuous or mixed depending upon the range of values they assume.
- A **discrete** random variable $x$ can take on a **countable** number of values $x_1, x_2, x_3, \ldots$ with probabilities $P\{x = x_i\}, i = 0, 1, 2, \ldots$
  
  - e.g., $\#$ of defective chips from a semiconductor wafer
- A **probability mass function (PMF)** $p_x(x_i)$ completely characterizes a discrete random variable. It is defined as
  
  $$p_x(x_i) = P\{x = x_i\}$$
  
  - Since $p_x(x_i)$ is a probability, it satisfies following properties
    
    $$0 \leq p_x(x_i) \leq 1, \quad \sum_i p_x(x_i) = \sum_i P\{x(\xi_i) = x_i | \xi_i \in \Omega\} = 1$$
Continuous Random Variables

• A continuous random variable $x$ takes values in a continuous set of numbers. The range of $x$ may include the whole real line or an interval thereof.

• Continuous random variables model many real life phenomena that include file download time on Internet, voltage across a resistor, and phase of a carrier signal produced by a radio transmitter.

• Therefore, we can not use the PMF for a continuous random variable. Instead we shall use the cumulative distribution function which serves as an appropriate probability measure for any random variable.
Example

• See notes DD1
The cumulative distribution function (CDF), $F_x(x)$, of a random variable $x$ is defined as

$$F_x(x) = P\{x \leq x\}$$

For any real number $x$, the CDF measures the probability that the random variable $x$ is no larger than $x$

- (a) $0 \leq F_x(x) \leq 1$

- (b) $\lim_{x \to -\infty} F_x(x) = 0$ and $\lim_{x \to \infty} F_x(x) = 1$

- (c) $P\{a < x \leq b\} = F_x(b) - F_x(a)$

- (d) $F_x(x)$ is nondecreasing
A probability density function (PDF), \( f_x(x) \), of a continuous random variable \( x \) is derivative of its CDF. That is:

\[
f_x(x) = \frac{dF_x(x)}{dx}
\]

The CDF of a continuous random variable \( x \) is integral of its PDF:

\[
F_x(a) = \int_{-\infty}^{a} f_x(x) \, dx
\]

- (a) \( f_x(x) \geq 0 \)
- (b) \( \int_{-\infty}^{\infty} f_x(x) \, dx = 1 \)
- (c) \( \int_{a}^{b} f_x(x) \, dx = P\{a < x \leq b\} \)

→ PDF is a continuous random variable is a function which can be integrated to obtain the probability that the random variable takes a value in a given interval.
Example

• CC1- See notes

The PDF of a random variable is given by

\[ f_x(x) = \begin{cases} 
  Ce^{-x}, & x \geq 0 \\
  0, & \text{otherwise}
\end{cases} \]

Find

a. The constant C
b. The CDF \( F_x(x) \)
c. \( P\{0 < x \leq 5\} \)
d. \( P\{-3 < x \leq 3\} \)
Common Discrete RVs

- Uniform
- Bernoulli
- Binomial
- Poisson
Uniform RV

- Totally Random – Equally likely events:

\[ P\{x = k\} = \frac{1}{M}, \quad k = 0, 1, 2, \ldots, M-1 \]

Its PMF can be

\[ p_x(x) = \begin{cases} 1/M, & k = 0, 1, 2, \ldots, M-1 \\ 0, & \text{otherwise} \end{cases} \]
Bernoulli Random Variable

- Binary Random variable where $0 < p < 1$
- Bernoulli random variables are used to model random experiments whose outcomes are binary
  - For example, whether a bit is received in error, or whether a packet is dropped by a congested router

$$P\{x = 1\} = p$$
$$P\{x = 0\} = 1 - p$$

Its PMF can be written

$$p_x(x) = \begin{cases} p, & x = 1 \\ 1 - p, & x = 0 \end{cases}$$
Binomial Random Variable

- Binomial random variables model the number of successes in a sequence of \( n \) independent trials of a random experiment, each of which yields success with probability \( p \).
- \( X \) RV is a binomial random variable if its PMF is of the form

\[
p_X(k) = P\{X = k\} = P\{k \text{ success in } n \text{ trials}\} = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots, n
\]

Remember: Combination Example: Picking a team of 3 people from a group of 10. \( C(10,3) = \frac{10!}{(7! \times 3!)} \)
The Poisson random variable $X$ models the number of events $(k)$ occurring in any interval $(t_o, t_o + \tau)$ if the occurrence of these events, at an average rate $\lambda$, is independent of $t_o$ and depends only on the length of interval $\tau$.

It is common in the literature to refer to the occurrence of a Poisson event as an arrival.

$X$ is a Poisson random variable if its PMF is of the form

$$p_X(k) = P(X = k) = P\{k \text{ arrivals in interval } \tau\} = e^{-\lambda \tau} \frac{(\lambda \tau)^k}{k!}, \quad k = 0, 1, 2, \ldots, \infty$$

where $\lambda = \text{average arrival rate}$.
Examples

• AA1
• BB1
Common Continuous Random Variables

- Here we introduce three important continuous random variables:
  - Uniform
  - Gaussian
  - Exponential
  - Poisson
  - Rayleigh
Uniform Random Variable

- $x$ is a uniform random variable if its PDF is given by

$$f_x(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b, \\ 0, & \text{otherwise}. \end{cases}$$

- The uniform random variable is a good model when each outcome of a random experiment is equally likely, and constrained to lie in the interval $[b, a], b > a$. 
Gaussian or Normal Random Variable

- $x$ is a normal or Gaussian random variable if its PDF is given by
  
  $$f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-m_x)^2}{2\sigma_x^2}}$$

- Characterized by mean $m_x$ and variance $\sigma_x^2$
  
  - $\sigma_x$ called the standard deviation

- A Gaussian random variable with mean $m_x$ and variance $\sigma_x^2$ is denoted by $\mathcal{N}(m_x, \sigma_x^2)$

- It is most frequently used random variable in the analysis and modeling of communication systems.
Gaussian or Normal Random Variable (contd)

- The CDF $F_x(x)$ of the Gaussian random variable $x$ is given by

$$F_x(x) = P\{x \leq x\} = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi} \sigma_x^2} e^{-\frac{(t-m_x)^2}{2\sigma_x^2}} \, dt$$

- There is no closed form solution for the integral on the right hand side. However, it can be written in terms of the $Q$-function as

$$F_x(x) = 1 - Q \left( \frac{x - m_x}{\sigma_x} \right) = Q \left( \frac{m_x - x}{\sigma_x} \right)$$

where

$$Q(a) = P\{x > a\} = \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} e^{-y^2/2} \, dy$$

Using Q-Function table $Q(a)$ can be found!

→ Next
Gaussian or Normal Random Variable (properties)

• Remember:
  – Q-Function is the area under standard normal RV

• Important Properties:

\[ Q(-x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^{\infty} e^{-y^2/2} dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy - \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-y^2/2} dy = 1 - Q(x) \]

\[ 1 - Q\left(\frac{x - m_x}{\sigma_x}\right) = Q\left(\frac{m_x - x}{\sigma_x}\right) \]

• Integrals for Q(z) cannot be evaluated in closed form. However, for large values of z, very good closed-form approximations can be obtained, and for small values of z, numerical integration techniques can be applied easily.
Table of Q-Function

Assuming SD = 1 and mean is 0

Table 1: Values of $Q(x)$ for $0 \leq x \leq 9$

<table>
<thead>
<tr>
<th></th>
<th>$Q(x)$</th>
<th></th>
<th>$Q(x)$</th>
<th></th>
<th>$Q(x)$</th>
<th></th>
<th>$Q(x)$</th>
<th></th>
<th>$Q(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.500000</td>
<td>2.30</td>
<td>0.010724</td>
<td>4.55</td>
<td>2.6823x10^{-6}</td>
<td>6.80</td>
<td>5.23110^{-12}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.05</td>
<td>0.48006</td>
<td>2.35</td>
<td>0.009387</td>
<td>4.60</td>
<td>2.1125x10^{-6}</td>
<td>6.90</td>
<td>3.6925x10^{-12}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.10</td>
<td>0.46017</td>
<td>2.40</td>
<td>0.008197</td>
<td>4.65</td>
<td>1.6597x10^{-6}</td>
<td>6.95</td>
<td>3.9121x10^{-12}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.15</td>
<td>0.44038</td>
<td>2.45</td>
<td>0.007142</td>
<td>4.70</td>
<td>1.3008x10^{-6}</td>
<td>7.00</td>
<td>4.0210x10^{-12}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.20</td>
<td>0.42074</td>
<td>2.50</td>
<td>0.006209</td>
<td>4.75</td>
<td>1.0171x10^{-6}</td>
<td>7.05</td>
<td>8.9439x10^{-12}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>0.40129</td>
<td>2.55</td>
<td>0.005386</td>
<td>4.80</td>
<td>7.9333x10^{-7}</td>
<td>7.10</td>
<td>6.2378x10^{-13}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.30</td>
<td>0.38209</td>
<td>2.60</td>
<td>0.004661</td>
<td>4.85</td>
<td>6.1731x10^{-7}</td>
<td>7.15</td>
<td>4.3389x10^{-13}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.35</td>
<td>0.36317</td>
<td>2.65</td>
<td>0.004024</td>
<td>4.90</td>
<td>4.7918x10^{-7}</td>
<td>7.20</td>
<td>3.0106x10^{-13}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.40</td>
<td>0.34458</td>
<td>2.70</td>
<td>0.003467</td>
<td>4.95</td>
<td>3.7107x10^{-7}</td>
<td>7.25</td>
<td>2.0839x10^{-13}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.45</td>
<td>0.32636</td>
<td>2.75</td>
<td>0.002979</td>
<td>5.00</td>
<td>2.8665x10^{-7}</td>
<td>7.30</td>
<td>1.4388x10^{-13}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>0.30854</td>
<td>2.80</td>
<td>0.002555</td>
<td>5.05</td>
<td>2.2091x10^{-7}</td>
<td>7.35</td>
<td>9.9103x10^{-14}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.55</td>
<td>0.29116</td>
<td>2.85</td>
<td>0.002186</td>
<td>5.10</td>
<td>1.6983x10^{-7}</td>
<td>7.40</td>
<td>6.8092x10^{-14}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.60</td>
<td>0.27425</td>
<td>2.90</td>
<td>0.001865</td>
<td>5.15</td>
<td>1.3024x10^{-7}</td>
<td>7.45</td>
<td>4.6671x10^{-14}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.65</td>
<td>0.25785</td>
<td>2.95</td>
<td>0.001589</td>
<td>5.20</td>
<td>9.9644x10^{-8}</td>
<td>7.50</td>
<td>3.1909x10^{-14}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.70</td>
<td>0.24196</td>
<td>3.00</td>
<td>0.001349</td>
<td>5.25</td>
<td>7.6050x10^{-8}</td>
<td>7.55</td>
<td>2.1763x10^{-14}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>0.22663</td>
<td>3.05</td>
<td>0.001142</td>
<td>5.30</td>
<td>5.7901x10^{-8}</td>
<td>7.60</td>
<td>1.4807x10^{-14}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.80</td>
<td>0.21186</td>
<td>3.10</td>
<td>0.000967</td>
<td>5.35</td>
<td>4.3977x10^{-8}</td>
<td>7.65</td>
<td>1.0049x10^{-14}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.85</td>
<td>0.19766</td>
<td>3.15</td>
<td>0.000816</td>
<td>5.40</td>
<td>3.3322x10^{-8}</td>
<td>7.70</td>
<td>6.8033x10^{-15}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.90</td>
<td>0.18406</td>
<td>3.20</td>
<td>0.000687</td>
<td>5.45</td>
<td>2.5185x10^{-8}</td>
<td>7.75</td>
<td>4.5945x10^{-15}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.95</td>
<td>0.17106</td>
<td>3.25</td>
<td>0.000577</td>
<td>5.50</td>
<td>1.8999x10^{-8}</td>
<td>7.80</td>
<td>3.0954x10^{-15}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>0.15866</td>
<td>3.30</td>
<td>0.000483</td>
<td>5.55</td>
<td>1.4283x10^{-8}</td>
<td>7.85</td>
<td>2.0802x10^{-15}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.05</td>
<td>0.14686</td>
<td>3.35</td>
<td>0.000404</td>
<td>5.60</td>
<td>1.0718x10^{-8}</td>
<td>7.90</td>
<td>1.3945x10^{-15}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.10</td>
<td>0.13567</td>
<td>3.40</td>
<td>0.000336</td>
<td>5.65</td>
<td>8.0224x10^{-9}</td>
<td>7.95</td>
<td>9.3256x10^{-16}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.15</td>
<td>0.12507</td>
<td>3.45</td>
<td>0.000280</td>
<td>5.70</td>
<td>5.9904x10^{-9}</td>
<td>8.00</td>
<td>6.2210x10^{-16}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.20</td>
<td>0.11507</td>
<td>3.50</td>
<td>0.000233</td>
<td>5.75</td>
<td>4.6422x10^{-9}</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example – Gaussian Distribution

A Gaussian random variable $x$ has the probability density function

$$f_x(x) = \frac{1}{\sqrt{30\pi}} \exp\left[-\frac{(x - 12)^2}{30}\right]$$

Express the following probabilities in terms of the $Q$-function:

a. $P(x \leq 11)$
b. $P(10 < x \leq 12)$
c. $P(11 < x \leq 13)$
d. $P(9 < x \leq 12)$
Example – Gaussian Distribution

A Gaussian random variable \( x \) has the probability density function

\[
f_x(x) = \frac{1}{\sqrt{30\pi}} \exp\left[-\frac{(x - 12)^2}{30}\right]
\]

Express the following probabilities in terms of the \( Q \)-function:

a. \( P(x \leq 11) \)
b. \( P(10 < x \leq 12) \)
c. \( P(11 < x \leq 13) \)
d. \( P(9 < x \leq 12) \)

Solution:

a. \( P(x \leq 11) = Q\left(\frac{12 - 11}{\sqrt{15}}\right) = Q\left(\frac{1}{\sqrt{15}}\right) \)

b. \( P(10 < x \leq 12) = P\{x \leq 12\} - P\{x \leq 10\} = Q(0) - Q\left(\frac{2}{\sqrt{15}}\right) \)

c. \( P(11 < x \leq 13) = Q\left(-\frac{1}{\sqrt{15}}\right) - Q\left(\frac{1}{\sqrt{15}}\right) = 1 - 2Q\left(\frac{1}{\sqrt{15}}\right) \)

d. \( P(9 < x \leq 12) = Q(0) - Q\left(\frac{3}{\sqrt{15}}\right) = 0.5 - Q\left(\frac{3}{\sqrt{15}}\right) \)

Use table to find the actual values.
Exponential Random Variable

- $X$ is an exponential random variable if its PDF is given by
  
  $f_X(x) = \begin{cases} 
  \lambda e^{-\lambda x}, & x \geq 0, \\
  0, & \text{otherwise}. 
  \end{cases}$
  
  where $\lambda > 0$

- For $x \geq 0$,

  $F_X(x) = P\{X \leq x\} = \int_0^x \lambda e^{-\lambda t} dt = \int_0^x \lambda e^{-\lambda x} dx = -e^{-\lambda t} \bigg|_0^x = 1 - e^{-\lambda x}$

- The exponential random variable is frequently used to model lifetimes (e.g., duration of a phone call) or waiting times (e.g., until some event happens).
<table>
<thead>
<tr>
<th>Name of Distribution</th>
<th>Type</th>
<th>Sketch of PDF</th>
<th>Cumulative Distribution Function (CDF)</th>
<th>Probability Density Function (PDF)</th>
<th>Mean</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Binomial</td>
<td>Discrete</td>
<td>[ F(x) = \sum_{k=0}^{n} P(k) ]</td>
<td>[ f(x) = \sum_{k=0}^{a} P(k) \delta(x - k) ]</td>
<td>np [ \text{and} ] np(1 - p)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Poisson</td>
<td>Discrete</td>
<td>[ F(x) = \sum_{k=0}^{\infty} P(k) ]</td>
<td>[ f(x) = \sum_{k=0}^{\infty} P(k) \delta(xk) ]</td>
<td>[ \lambda ] [ \text{and} ] [ \lambda ]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Uniform</td>
<td>Continuous</td>
<td>[ F(x) = \begin{cases} 0, &amp; a &lt; \left( \frac{2m - A}{2} \right) \ \frac{1}{A} \left( a - \left( \frac{2m - A}{2} \right) \right), &amp;</td>
<td>a - m</td>
<td>\leq \frac{A}{2} \ 1, &amp; a \geq \left( \frac{2m - A}{2} \right) \end{cases} ]</td>
<td>[ f(x) = \begin{cases} 0, &amp; x &lt; \left( \frac{2m - A}{2} \right) \ \frac{1}{A}, &amp;</td>
<td>x - m</td>
</tr>
<tr>
<td>Gaussian</td>
<td>Continuous</td>
<td>[ F(x) = Q\left( \frac{m - a}{\sigma} \right) ]</td>
<td>[ f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\left(x - m\right)^2/2\sigma^2\right] ]</td>
<td>m [ \sigma^2 ]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sinusoidal</td>
<td>Continuous</td>
<td>[ F(x) = \begin{cases} 0, &amp; a \leq -A \ \frac{1}{A} \left[ \frac{\pi}{2} + \sin^{-1} \left( \frac{a}{A} \right) \right], &amp;</td>
<td>a</td>
<td>\leq A \ 1, &amp; a \geq A \end{cases} ]</td>
<td>[ f(x) = \begin{cases} 0, &amp; x &lt; -A \ \frac{1}{\pi\sqrt{A^2 - x^2}}, &amp;</td>
<td>x</td>
</tr>
</tbody>
</table>
Example

Assume the phase offset between the transmitter and the receiver is modeled by a random variable \( \Theta \) that is uniformly distributed between \([-\pi , \pi]\).

a. \( P(\theta \leq 0) \)
b. \( P(\theta \leq \pi/4) \)

This is continuous RV \( \rightarrow \) Find \( f_x(\theta) \)
Example

Assume the phase offset between the transmitter and the receiver is modeled by a random variable Theta that is uniformly distributed between \([-\pi, \pi]\). Find

a. \(P\{\theta \leq 0\}\)
b. \(P\{\theta \leq \pi/4\}\)

Solution

Because \(\theta\) is uniformly distributed between \([-\pi, \pi]\), its PDF is given as follows:

\[
f_{\theta}(\theta) = \begin{cases} 
  \frac{1}{2\pi}, & -\pi \leq \theta \leq \pi \\ 
  0, & \text{otherwise}
\end{cases}
\]

a. \(P\{\theta \leq 0\} = \int_{-\infty}^{0} f_{\theta}(\theta) d\theta = \int_{-\pi}^{0} \frac{1}{2\pi} d\theta = \frac{\pi}{2\pi} = \frac{1}{2}\)

b. \(P\{\theta \leq \pi/4\} = \int_{-\infty}^{\pi/4} f_{\theta}(\theta) d\theta = \int_{-\pi}^{\pi/4} \frac{1}{2\pi} d\theta = \frac{\pi}{4} + \frac{\pi}{2\pi} = \frac{5}{8}\)
Poisson Random Variable

- The Poisson random variable $\lambda$ models the number of events ($k$) occurring in any interval $(t_o, t_o + \tau)$ if the occurrence of these events, at an average rate $\lambda$, is independent of $t_o$ and depends only on the length of interval $\tau$.

- It is common in the literature to refer to the occurrence of a Poisson event as an arrival.

- $\lambda$ is a Poisson random variable if its PMF is of the form

$$p_x(k) = P(x = k) = P\{k \text{ arrivals in interval } \tau\} = e^{-\lambda \tau} \frac{\lambda^k \tau^k}{k!} \quad k = 0,1,2,\ldots,\infty$$

where $\lambda = \text{average arrival rate}$.
Statistics of RV

- Finding behaviors using certain averages
  - Mean, Variance, Standard Deviation, Moments, Central Moments, etc.

- The *expected value* or *mean* of a continuous random variable $x$ is defined as
  
  $$m_x = \bar{x} = E\{x\} = \int_{-\infty}^{+\infty} x f_x(x) \, dx$$

- The expected value of a random variable represents its average value in a very large number of trials.

- The mean of the function $y = g(x)$ is
  
  $$\bar{g(x)} = E\{g(x)\} = \int_{-\infty}^{+\infty} g(x) f_x(x) \, dx$$

- The variance $Var(x)$ of a random variable $x$ is defined as
  
  $$Var(x) = \sigma_x^2 = E\{(x - m_x)^2\} = \int_{-\infty}^{+\infty} (x - m_x)^2 f_x(x) \, dx \geq 0$$

  Describes the spread of its PDF around the expected value.
Statistics of RV (cont.)

- Variance
- Root-Mean-Square

\[
Var(x) = \int_{-\infty}^{\infty} (x^2 - 2mx + m^2)f_x(x)\,dx \\
= \int_{-\infty}^{\infty} x^2f_x(x)\,dx - 2m\int_{-\infty}^{\infty} xf_x(x)\,dx + m^2 \\
= E\{x^2\} - m_x^2 = \bar{x}^2 - \bar{x}^2
\]

- Note that when mean is zero variance is the same as RMS:

\[
Var(x) = E\{x^2\}
\]

- Standard Deviation of a RV is

\[
\sigma_x = \sqrt{Var(x)}
\]
Moments of a RV

- Expected value $E\{x\}$ is the First Moment of a RV
- RMS value $E\{x^2\}$ is the Second Moment of a RV
- The $n^{th}$ moment of a real-valued random variable $x$ is
  \[ E\{x^n\} = \int_{-\infty}^{\infty} x^n f_x(x) \, dx \]

- The $n^{th}$ central moment of a real-valued random variable $x$ is
  \[ E\{(x - m_x)^n\} = \int_{-\infty}^{\infty} (x - m_x)^n f_x(x) \, dx \]

- Hence the variance $\text{Var}(x)$ is the second central moment of $x$
  \[ \text{Var}(x) = E\{x^2\} - m_x^2 = \bar{x}^2 - \bar{x}^2 \]
Example 1 – Mean & Variance

Find the mean and variance of exponential random variable \( x \) with PDF

\[
f_x(x) = \begin{cases} 
\lambda e^{-\lambda x}, & x \geq 0 \\
0, & \text{otherwise}
\end{cases}
\]

where \( \lambda > 0 \).
Find the mean and variance of exponential random variable $x$ with PDF

$$f_x(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

where $\lambda > 0$.

- The $n$th moment (integ. by part):
  
  Thus, for $n=0 \rightarrow E\{x^0\}=1$ (*)

For $n = 1$, we have $E\{x\} = \bar{x} = \frac{1}{\lambda} E\{x^0\} = \frac{1}{\lambda}$

For $n = 2$, we have $E\{x^2\} = \bar{x}^2 = \frac{2}{\lambda} E\{x\} = \frac{2}{\lambda^2} \frac{1}{\lambda} = \frac{2}{\lambda^2}$

$$Var(x) = \bar{x}^2 - \bar{x}^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} = \text{Second moment – first moment square!}$$

Integration Table (number 57 – Ingration by part)

\[ (*) \int e^{ax} \, dx = \frac{1}{a} e^{ax} \]
Paired Random Variables - CDF

- Random experiments where the outcomes are described by a pair of random variables $x$ and $y$
  - Example: the cumulative GPA ($x$) and SAT score ($y$) of a graduating high school senior in CA!
  - Signal $x$ emitted by a radio transmitter and the corresponding signal $y$ that eventually arrives at the receiver

- The joint cumulative distribution function (CDF) of two random variables $x$ and $y$ is defined as
  $$F_{xy}(x, y) = P\{x \leq x, y \leq y\}$$

- Note that $F_{xy}(x, y)$ measures the probability of event $A = \{\xi \in \Omega : x(\xi) \leq x, y(\xi) \leq y\}$

Example:

$$F_{xy}(0.1, 1.5) = \int_{0}^{0.1} \int_{0}^{1.5} f_{xy}(x, y) \, dx \, dy$$

Properties:

(a) $0 \leq F_{xy}(x, y) \leq 1$
(b) $F_{xy}(\infty, \infty) = 1$
(c) $F_{xy}(x, -\infty) = F_{xy}(-\infty, y) = 0$
(d) $F_{xy}(x, y)$ is nondecreasing
Paired Random Variables - PDF

Joint Probability Density Function

The joint probability density function, \( f_{xy}(x, y) \), of two random variables \( x \) and \( y \) is defined as

\[
f_{xy}(x, y) = \frac{\partial^2 F_{xy}(x, y)}{\partial x \partial y}
\]

\[\Rightarrow F_{xy}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{uv}(u, v) \, du \, dv\]

Properties:

(a) \( f_{xy}(x, y) \geq 0 \) for all \( (x, y) \)

(b) \[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{xy}(x, y) \, dx \, dy = F_{xy}(\infty, \infty) = 1\]

(c) For a rectangle \( \{a < x \leq b, c < y \leq d\} \) in \( x-y \) plane,

\[P\{a < x \leq b, c < y \leq d\} = \int_{a}^{b} \int_{c}^{d} f_{xy}(x, y) \, dx \, dy\]
**Paired Random Variables – Conditional PDF**

- The conditional PDF of random variable $x$ given $\{y = y\}$, denoted by $f_x(x|y)$, is defined as

$$f_x(x|y) = f_x(x|y = y) = \frac{f_{x,y}(x,y)}{f_y(y)}, \quad f_y(y) > 0$$

- Note that for each $y$ with $f_y(y) > 0$, the conditional PDF $f_x(x|y)$ provides a new probabilistic description of the random variable $x$.

- Similarly, we can define

$$f_y(y|x) = f_y(y|x = x) = \frac{f_{x,y}(x,y)}{f_x(x)}, \quad f_x(x) > 0$$

Note: It is possible to find $f_y(y)$ from $f_{x,y}(x,y)$ over the given range for $x$:

$$f_y(y) = \int_{0}^{6} f_{x,y}(x,y)dx$$
Two random variables \( x \) and \( y \) are said to be statistically independent if

\[
F_{xy}(x, y) = P\{x \leq x, y \leq y\} = P\{x \leq x\} P\{y \leq y\} = F_x(x)F_y(y)
\]

Equivalently, for independent random variables

\[
f_{xy}(x, y) = f_x(x)f_y(y)
\]

For independent random variables,

\[
f_x(x | y) = \frac{f_{xy}(x, y)}{f_y(y)} = \frac{f_x(x)f_y(y)}{f_y(y)} = f_x(x)
\]

\[
f_y(y | x) = f_y(y)
\]

The PDF of \( x \) after knowledge of the event \( \{y = y\} \) same as its PDF before the knowledge.
Statistics of Paired RV

- Expected value of $x + y$
  \[ E\{x + y\} = E\{x\} + E\{y\} \]

- More generally, expectation is a linear operator
  \[ E\left\{\sum \alpha_i x_i\right\} = \sum \alpha_i E\{x_i\} \]

- Variance of $x + y$
  \[ Var(x + y) = Var(x) + Var(y) + 2E\{(x - m_x)(y - m_y)\} \]

- Covariance of $x$ and $y$
  \[ Cov(x, y) = E\{(x - m_x)(y - m_y)\} \]

  \[ \Rightarrow Var(x + y) = Var(x) + Var(y) + 2Cov(x, y) \]
Correlation and Covariance of Two RVs

- The correlation of two random variables $x$ and $y$ is defined as
  $$R_{xy} = E(xy)$$

- It is very simple exercise to prove that
  $$Cov(x, y) = E(xy) - E(x)E(y) = R_{xy} - m_x m_y$$

- $x$ and $y$ are called uncorrelated random variables if
  $$Cov(x, y) = 0$$
  $$\Rightarrow E(xy) = E(x)E(y)$$

- The correlation coefficient of two random variables $x$ and $y$ is defined as
  $$\rho_{xy} = \frac{Cov(x, y)}{\sigma_x \sigma_y}$$

  Corr. Corr is between 0 & 1
  If $CC = 0 \rightarrow$ two RVs are uncorrelated
  If $CC >= 0 \rightarrow$ two RVs are moving in the same direction
  If $CC < 0 \rightarrow$ two RVs are moving in different directions
Let $x_1, x_2, \ldots$ be $n$ independent, identically distributed random variables with finite mean and variance. We consider their scaled sum

$$z_n = \frac{\sum_{i=1}^{n} (x_i - m)}{\sigma \sqrt{n}} = \frac{s_n - nm}{\sigma \sqrt{n}}$$

- The CDF of $z_n$ converges to a Gaussian CDF as $n$ approaches $\infty$, independent of the distribution of random variables $x_n$.
- In a nutshell, the central limit theorem, states that the sum of almost any set of independent and randomly generated random variables rapidly converges to the Gaussian distribution $\lambda$.
- This explains why the Gaussian distribution arises so commonly in practice to reflect the additive effect of multiple random occurrences.
Example 2 – Joint PDF

The joint PDF of two random variables is

\[ f_{xy}(x, y) = \begin{cases} C(1 + xy), & 0 \leq x \leq 6, \ 0 \leq y \leq 5 \\ 0, & \text{otherwise} \end{cases} \]

Find the following:

a. The constant \( C \)
b. \( F_{xy}(0.1, 1.5) \)
c. \( f_{xy}(x, 3) \)
d. \( f_x(x|y) \)
Example 3 – Statistical Averages
• Later
References

• Leon W. Couch II, Digital and Analog Communication Systems, 8th edition, Pearson / Prentice, Chapter 6