Chapter 6

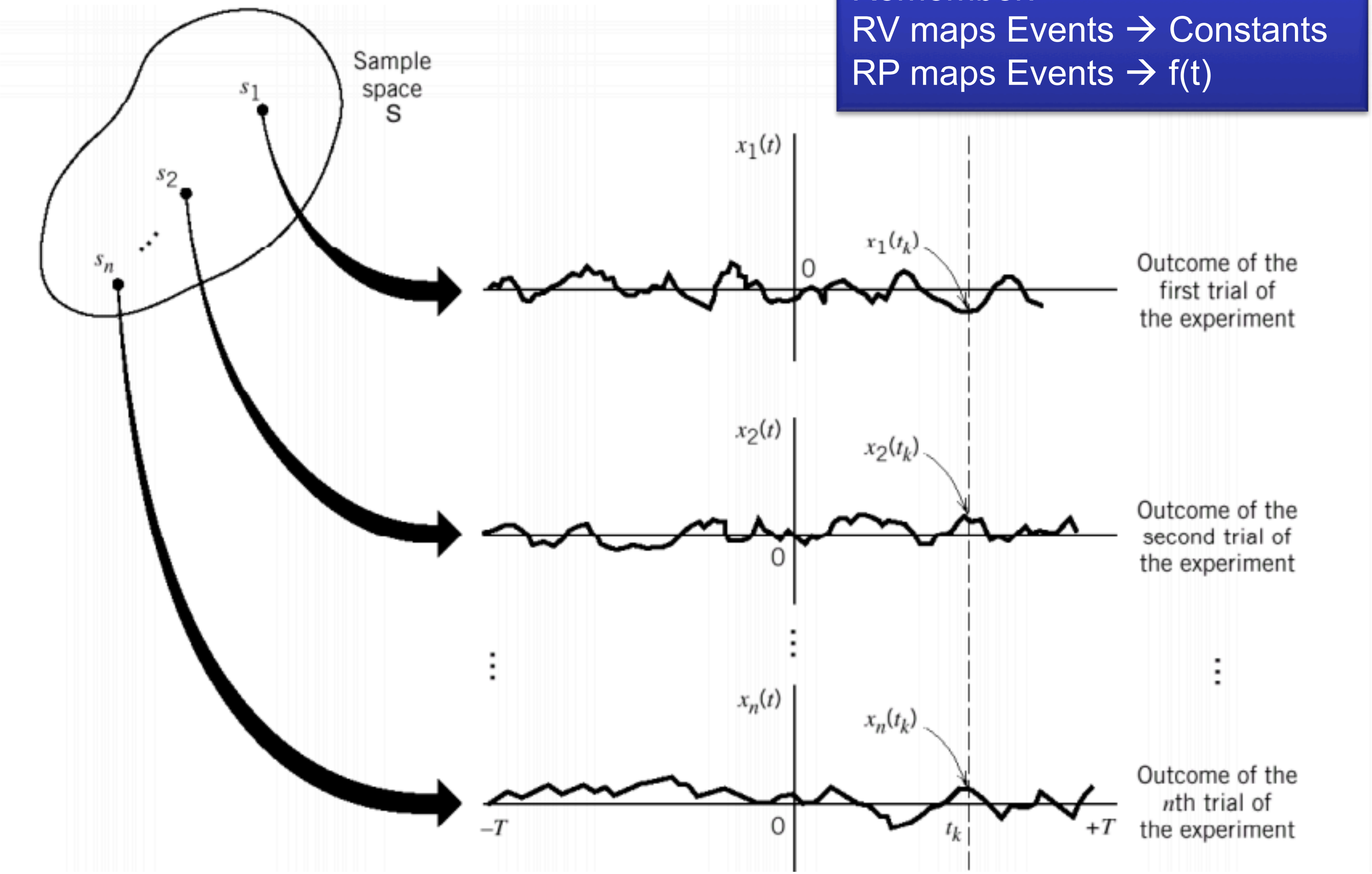
Random Processes

Random Process

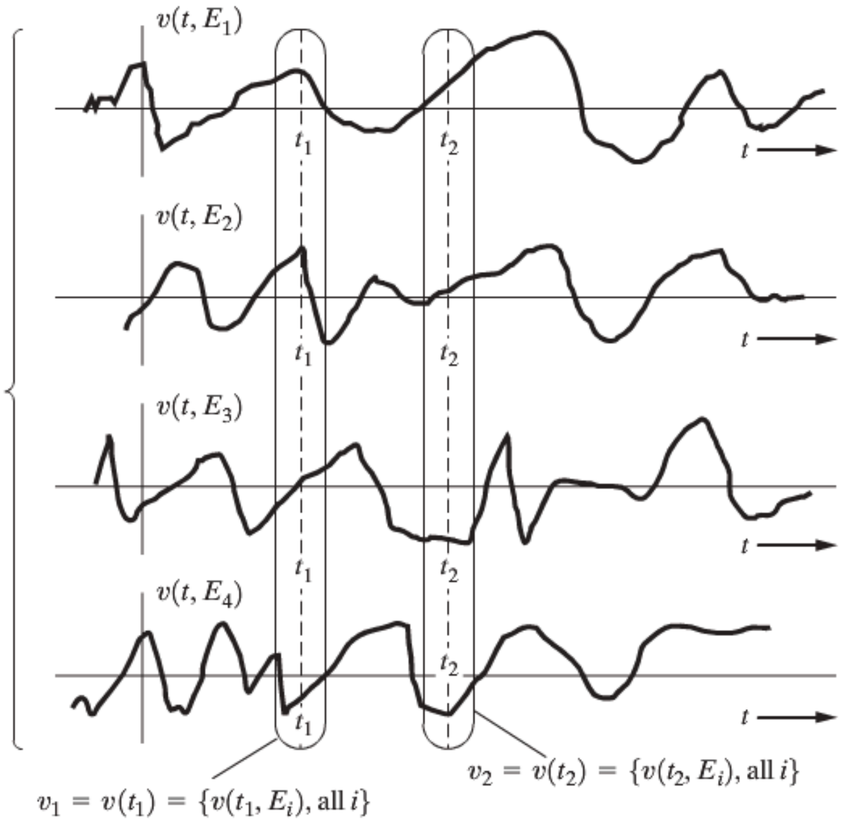
- A random process is a time-varying function that assigns the outcome of a random experiment to each time instant: X(t).
- For a fixed (sample path): a random process is a time varying function, e.g., a signal.
  - For fixed t: a random process is a random variable.
- If one scans all possible outcomes of the underlying random experiment, we shall get an ensemble of signals.
- Random Process can be **continuous or discrete**
- Real random process also called stochastic process
  - Example: Noise source (Noise can often be modeled as a Gaussian random process.

An Ensemble of Signals

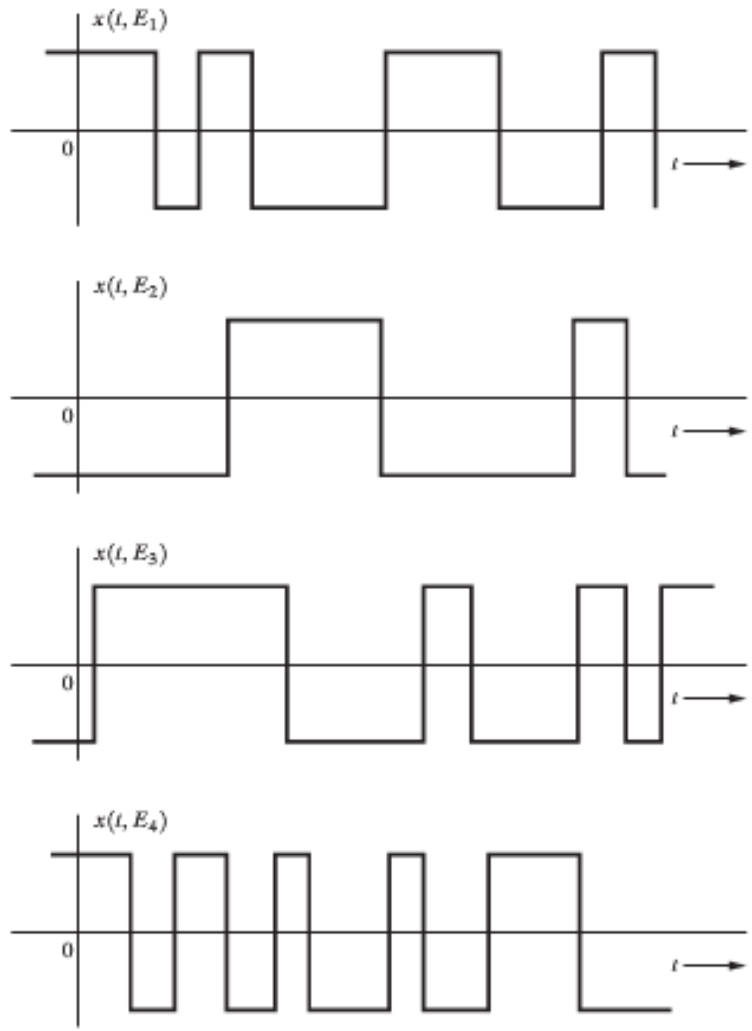
Remember:
RV maps Events → Constants
RP maps Events → f(t)



RP: Discrete and Continuous



The set of all possible sample functions {v(t, E i)} is called the ensemble and defines the random process v(t) that describes the noise source.



Sample functions of a binary random process.

RP Characterization

- Random variables $x_1, x_2, \ldots, x_n$ represent amplitudes of sample functions at t 5 $t_1, t_2, \ldots, t_n$.
  - A random process can, therefore, be viewed as a collection of an infinite number of random variables:

$$\text{joint PDF } f_x(x_1, x_2, \ldots, x_n, t_1, t_2, \ldots, t_n):$$

RP Characterization – First Order

- CDF

$$F_{\mathbf{x}}(x, t) = P\{\mathbf{x}(t) \leq x\}$$

- PDF

$$\boxed{f_{\mathbf{x}}(x, t)} = \frac{dF_{\mathbf{x}}(x, t)}{dx}$$

- Mean

$$m_{\mathbf{x}}(t) = \overline{\mathbf{x}(t)} = E\{\mathbf{x}(t)\} = \int_{-\infty}^{+\infty} x f_{\mathbf{x}}(x, t) dx$$

- Mean-Square

$$\overline{\mathbf{x}^2(t)} = E\{\mathbf{x}^2(t)\} = \int_{-\infty}^{+\infty} x^2 f_{\mathbf{x}}(x, t) dx$$

Statistics of a Random Process

- For fixed *t*: the random process becomes a random variable, with mean

$$\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x\, f_X(x;t)dx$$

  – In general, the mean is a function of t.

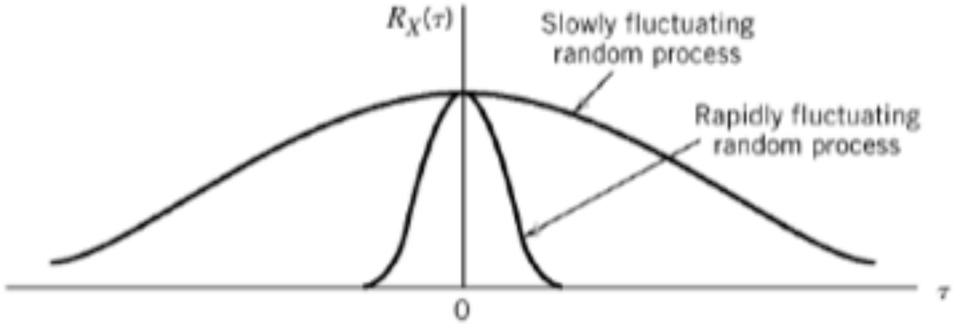
- Autocorrelation function

$$R_X(t_1,t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x,y;t_1,t_2)dxdy$$

  – In general, the autocorrelation function is a two-variable function.
  – It measures the correlation between two samples.

RP Characterization – Second Order

- The first order does not provide sufficient information as to how rapidly the RP is changing as a function of time→ We use second order estimation



RP Characterization – Second Order

- The first order does not provide sufficient information as to how rapidly the RP is changing as a function of time→ We use second order estimation

$$F_x(x_1, x_2, t_1, t_2) = P\{x(t_1) \le x_1, x(t_2) \le x_2\}$$

- CDF

$$f_x(x_1, x_2, t_1, t_2) = \frac{\partial^2 F_x(x_1, x_2, t_1, t_2)}{\partial x_1 \partial x_2}$$

- PDF

- Auto-correlation

(statistical average of the product of RVs)

$$R_x(t_1, t_2) = E\{x(t_1)x(t_2)\}$$

$$R_{xy}(t_1, t_2) = E\{x(t_1)y(t_2)\}$$

- Cross-Correlation

(measure of correlation between sample function amplitudes of processes x (t) and y (t) at time instants t 1 and t 2 , respectively)

Example

- Example A

Stationary RP

- We can characterize RP based on how their statistical properties change
- If the statistical properties of a RP don't change with time we call the RP stationary, then first-order does not depend on time:

$$f_x(x, t) = f_x(x)$$

- Strict-Sense Stationary:

$$m_x(t) = \overline{x(t)} = E\{x(t)\} = \int_{-\infty}^{\infty} x f_x(x) dx = \text{constant}$$

$$\overline{x^2(t)} = E\{x^2(t)\} = \int_{-\infty}^{\infty} x^2 f_x(x) dx = \text{constant}$$

First Order

$$f_x(x_1, x_2, t_1, t_2) = f_x(x_1, x_2, t_1 - t_2)$$

$$R_x(t_1, t_2) = E\{x(t_1)x(t_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_x(x_1, x_2, t_1 - t_2) dx_1 dx_2$$

$$= R_x(t_1 - t_2) = R_x(\tau)$$

Second Order

the second-order PDF of a stationary process is independent of the time origin and depends only on the time difference t 1 - t 2 .

Stationary RP

- We ca … tical prope…
- If the … ith time we call th … pend on time: … x)
- Strict-…

Because the conditions for the first- and second-order stationary are usually difficult **to verify in practice,** we define the concept of **wide-sense stationary** that represents a less stringent requirement.

$$m_x(t) = \overline{x(t)} = E\{x(t)\} = \int_{-\infty}^{\infty} x f_x(x)dx = \text{constant}$$

$$\overline{x^2(t)} = E\{x^2(t)\} = \int_{-\infty}^{\infty} x^2 f_x(x)dx = \text{constant}$$

First Order

$$f_x(x_1, x_2, t_1, t_2) = f_x(x_1, x_2, t_1 - t_2)$$

$$R_x(t_1, t_2) = E\{x(t_1)x(t_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2\, f_x(x_1, x_2, t_1 - t_2)dx_1dx_2$$

$$= R_x(t_1 - t_2) = R_x(\tau)$$

Second Order

the second-order PDF of a stationary process is independent of the time origin and depends only on the time difference t 1 - t 2 .

Wide-Sense Stationary RP

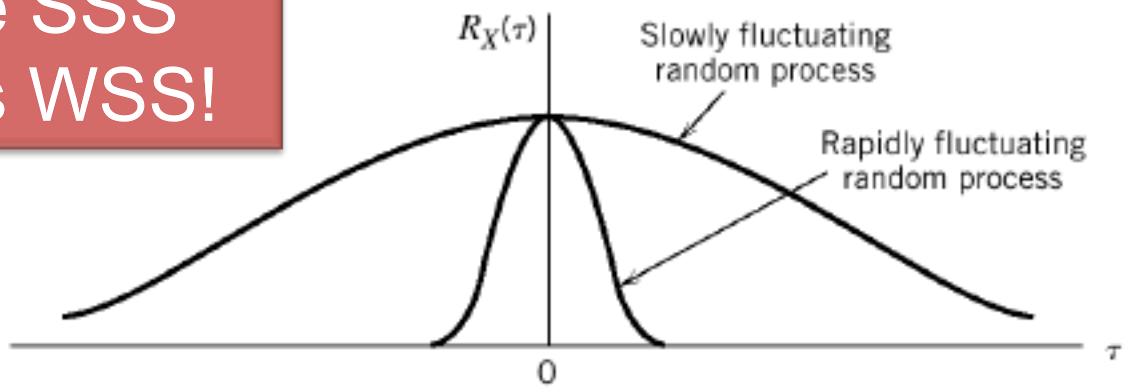
- A random process is (wide-sense) stationary if
  - Its mean does not depend on *t*

$$m_x(t) = \overline{x(t)} = E\{x(t)\} = \text{constant}$$

  - Its autocorrelation function only depends on time difference

$$R_X(t, t+\tau) = R_X(\tau)$$

Note that the SSS RP is always WSS!



- In communications, noise and message signals can often be modelled as stationary random processes.

WSS RP – Properties

For a WSS random process x (t), the autocorrelation function has the following important properties:

1. $R_x(0) = E\{x^2(t)\} = \overline{x^2(t)} \geq 0$

Thus $R_x(0)$ represents the total power of the random signal $x(t)$.

2. $R_x(\tau) = R_x(-\tau)$

3. $\lim_{|\tau| \to \infty} R_x(\tau) = \lim_{|\tau| \to \infty} E\{x(t)x(t+\tau)\} = E\{x(t)\}E\{x(t+\tau)\} = \overline{x(t)}^2$

For $|\tau|$ large, $R_x(\tau)$ represents the average or DC power of the random signal.

4. $|R_x(\tau)| \leq |R_x(0)|$ for all τ

Remember

- rth moment:

$$\overline{(x - x_0)^r} = \int_{-\infty}^{\infty} (x - x_0)^r f(x) dx$$

- Mean (first moment, Xo=0)

$$m \triangleq \overline{x} = \int_{-\infty}^{\infty} x f(x) dx$$

$$\sigma^2 = \overline{x^2} - 2(\overline{x})^2 + (\overline{x})^2$$

- Variance

second moment about the mean

$$\sigma^2 = \overline{(x - \overline{x})^2} = \int_{-\infty}^{\infty} (x - \overline{x})^2 f(x) dx$$

Prove this:

$$\sigma^2 = \overline{x^2} - 2(\overline{x})^2 + (\overline{x})^2$$

- Standard Deviation

Square-rood of variance Square-rood of variance

$$\sigma = \sqrt{\sigma^2} = \sqrt{\int_{-\infty}^{\infty} (x - \overline{x})^2 f(x) dx}$$

Relation Between Different Random Processes

- Uncorrelated

$$R_{xy}(t, t+\tau) = E\{x(t)y(t+\tau)\} = E\{x(t)\}E\{y(t+\tau)\}$$

$$Cov\big(x(t), y(t+\tau)\big) = E\{x(t)y(t+\tau)\} - E\{x(t)\}E\{y(t+\tau)\} = 0$$

cross-covariance

- Orthogonal

$$R_{xy}(\tau) = 0$$ =Cross-correlation

- Independent

if the set of random variables x (t 1), x (t 2), . . . , x (t n) is statistically independent of the set of random variables y(t’1), y(t’2), c, y(t’n) for any choice of t 1 , t 2 , . . . , t n and t’1, t’2,etc.

Ergodic RP

- The computation of statistical averages (e.g., mean and autocorrelation function) of a random process requires an ensemble of sample functions (data records) that may not always be feasible.
- In many real-life applications, it would be very convenient to calculate the averages from a single data record.
- This is possible in certain random processes called ergodic processes.

Ergodic RP

- The ergodic assumption implies that any sample function of the process takes all possible values in time with the same relative frequency that an ensemble will take at any given instant:

$$\overline{x(t)} = E\{x(t)\} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)dt = \langle x(t) \rangle$$

Ensemble function

Time Average

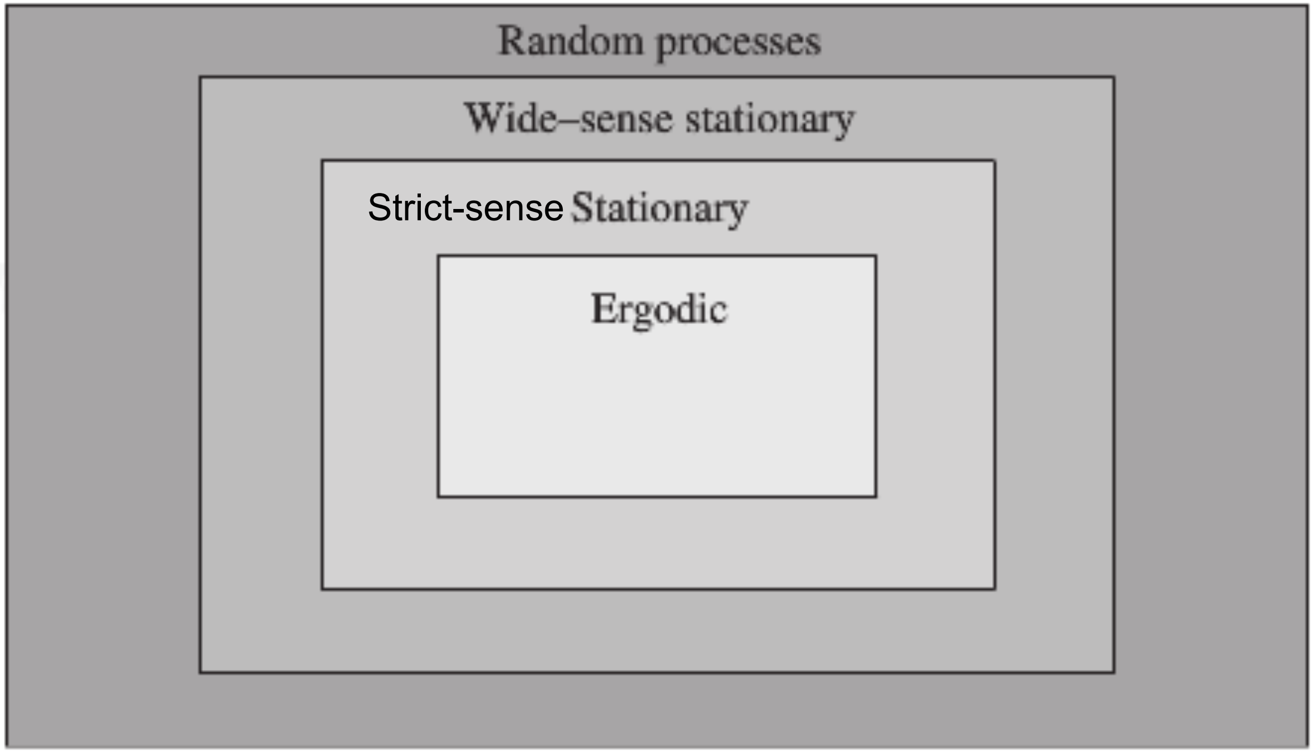
$$R_x(\tau) = E\{x(t)x(t+\tau)\} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x^*(t-\tau)dt = \mathcal{R}_x(\tau)$$

Where <x (t)> and **R**x(t) are time-average mean and autocorrelation function

Difficult to verify if a RP is Ergodic!
Because we have to verify the
ensemble averages and time averages of all orders!

Classification of Random Processes

- Summary:



Example B

Consider the following examples:

$$f(x) = \begin{cases} \dfrac{1}{\pi\sqrt{A^2 - x^2}}, & |x| \leq A \\ 0, & x \text{ elsewhere} \end{cases}$$

First order PDF →

Not a function of t →

PDF stationary process

First order PDF →

Is a function of t →

PDF is NOT stationary process

$$f(x) = \delta(x - A\ \sin(\omega_0 t + \theta_0))$$

Example C

- Show that sinusoidal wave with random phase

$$X(t) = A\cos(\omega_c t + \Theta)$$

with phase Θ uniformly distributed on [0,2π] is stationary.

Find mean
Find auto-correlation
Is it WSS RP?
Is it WSS periodic RP?

Example C

- Show that sinusoidal wave with random phase

$$X(t) = A\cos(\omega_c t + \Theta)$$

with phase Θ uniformly distributed on [0,2π] is stationary.

  – Mean is a constant:

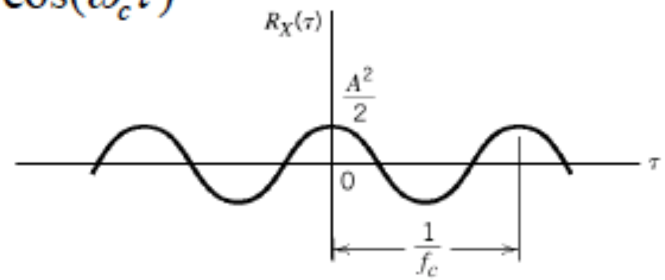
$$\mu_X(t) = E[X(t)] = \int_0^{2\pi} A\cos(\omega_c t + \theta)\frac{1}{2\pi}d\theta = 0$$

$$f_\Theta(\theta) = \frac{1}{2\pi}, \quad \theta \in [0, 2\pi]$$

  – Autocorrelation function only depends on the time difference:

$$\begin{aligned} R_X(t, t+\tau) &= E[X(t)X(t+\tau)] \\ &= E[A^2 \cos(\omega_c t + \Theta)\cos(\omega_c t + \omega_c \tau + \Theta)] \\ &= \frac{A^2}{2}E[\cos(2\omega_c t + \omega_c \tau + 2\Theta)] + \frac{A^2}{2}E[\cos(\omega_c \tau)] \\ &= \frac{A^2}{2}\int_0^{2\pi}\cos(2\omega_c t + \omega_c \tau + 2\theta)\frac{1}{2\pi}d\theta + \frac{A^2}{2}\cos(\omega_c \tau) \end{aligned}$$

$$R_X(\tau) = \frac{A^2}{2}\cos(\omega_c \tau)$$



Examples

- Example D – Ergodic